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Finite strain plasticity models revealed by symmetries and integrating factors: The case of Dafalias spin model

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ABSTRACT

We consider a rigid plastic constitutive model with linear kinematic hardening, relying on the concept of constitutive spin introduced by Dafalias (1985a,b) to describe the evolution of the orientation of the material texture. A more general writing of the constitutive spin using representation theorems for second order tensors is proposed, involving arbitrary functions of the tensor invariants. The computation of continuous symmetries and integrating factors of the resulting system of differential equations leads to a classification of cases, in terms of the constitutive functions, focusing on simple planar shear. Exact and numerical solutions for stress versus time are obtained for some objective rates. The comparison of the evolution of the integrated stress components allows drawing some conclusion as to the more suitable objective rates. Dynamical invariants computed in terms of the components of the back stress tensors and of the shear strain allow to directly evaluate the dynamical response of the material in terms of the phase portrait in the space of independent components of the back stress tensor. Fundamental principles of irreversible thermodynamics are used as a filtering mechanism for the constitutive models revealed by symmetries and invariants, leading to the choice of constitutive models that satisfy all proposed criteria. These models involve a non-linear dependence of the plastic spin on the back stress.

1. Introduction

Hypoelastic models and finite strain plasticity models involve proper objective rates for both the stress and back stress when kinematic hardening is concerned. The stress is an important field that intervenes together with its rate both in the elastic part of an elastoplastic model (when a hypoelastic formulation is used), in the definition of a yield criterion in stress space, and in the evolution laws for the plastic deformation and tensorial like variables such as the back stress in kinematic hardening models. Anisotropic material behavior (especially in sheet metals and more generally in polycrystalline aggregates) can be traced back to texture and microstructure development (Barlat et al., 2003). Different strategies have been considered in order to model elastic and plastic anisotropic material behavior, such as structure tensors (Boehler, 2006; Liu, 1982; Zhang and Rychlewski, 1990), the concept of plastic spin (Dafalias, 1998) or deformation-like internal variables associated to kinematic hardening (Svendsen, 1998, 2001). The connection between the orientational kinematics of the continuum and its substructure has been given by the plastic spin concept, introduced in Dafalias (1985b) to reflect the difference of the material and director vectors in the theory initially developed in Mandel (1971). The plastic spin is related to the

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evolution of the orientation of the material microstructure, a rather clear concept, whereas the directors are associated to the material substructure, which becomes more evasive when using a more macroscopic approach in terms of tensorial internal variables. As underlined in Dafalias (1998), the plastic spin has to be dissociated from the general kinematics of the deformation, and has to be independently described as a representation of the orientation of the material microstructure as part of the evolution laws for the other internal variables. Hence, a plasticity model should include a model for the orientational kinematics in terms of co-rotational rates of microstructure variables via the notion of spin, together with the evolution of the internal state variables (for instance the back stress). In finite deformation analysis, the constitutive model must involve the corotational rates (Lubarda, 2001) for the material response to be independent of rigid body rotations. Different spin tensors have been defined associated to different corotational rates. Ghavam and Naghdabadi (2007) proposed a general method to obtain the spin tensors associated with corotational rates based on deformation. However, plastic deformation dependent spin tensors are preferred; Ghavam and Naghdabadi (2011) introduce a plastic deformation dependent spin tensor and its associated corotational rate. The proposed governing equations are solved for the simple shear problem. Dafalias (1985b) makes a distinction between the plastic spin $\mathbf{W}^p$ arising as the plastic part of the total spin $\mathbf{W}$ and the constitutive spin $\boldsymbol{\omega}$ forming the definition of the co-rotational rate of stress. In general, he writes the plastic spin as

$$\mathbf{W}^p = \langle\lambda\rangle\boldsymbol{\Omega}^p(\boldsymbol{\sigma},\mathbf{s}), \tag{1}$$

where $\langle\lambda\rangle$ is the positive part of the plastic multiplier, $\boldsymbol{\sigma}$ is the Cauchy stress, and $\mathbf{s}$ is an internal state variable such as the plastic strain rate. In parallel to this, extensive research in the field of plasticity has been devoted to the topic related to plastic spin on how to eliminate spurious stress oscillations appearing in the simulation of fixed-end torsion, with the many available definitions of objective rates that can be classified as co-rotational or non-corotational, otherwise coined spinning and non-spinning respectively (Liu, 2004). The same author has shown that about 10 of the most known objective stress rates can be condensed into a generic format involving the Poisson bracket of a second order kinematic tensor $\mathbf{B}$ (taking a different definition according to the considered model) as

$$\hat{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}} - 2[\mathbf{B}\boldsymbol{\tau}], \tag{2}$$

where $\boldsymbol{\tau} = J\boldsymbol{\sigma}$ denotes the Kirchhoff stress, the dot stands for the time derivative, $\hat{\boldsymbol{\tau}}$ denotes the objective stress rate of the Kirchhoff stress, and $J \equiv \det\mathbf{F}$ is the Jacobian of the tangent mapping. The bracket $[\cdot]$ in (2) denotes symmetrization of the enclosed tensor. The rate is said to be co-rotational when the $\mathbf{B}$ tensor in (2) is antisymmetrical; co-rotational rates have the advantage of decoupling the volumetric and deviator constitutive equations, which is not the case for the non co-rotational rates. Note that as shown in Arghavani et al. (2011), the logarithmic strain introduced by Hencky in 1928 has the property that its corotational rate is the strain rate tensor. Although the objective Kirchhoff stress rate (2) is linear in the stress, its formulation in terms of the stress deviator instead leads to a non-linear representation of the adopted constitutive model – hypoelastic combined with perfect plasticity, as reflected by Eq. (45) in Liu (2004). This is in line with the representation theorems for tensorial valued functions with a tensorial argument, as exposed next considering the constitutive models for kinematic hardening: the representation theorem of Wang (1970) leads indeed to a general writing of the functional dependence of the plastic spin in (1) as

$$\boldsymbol{\Omega}^p = c_1[\mathbf{s},\boldsymbol{\sigma}] + c_2\left[\mathbf{s},\boldsymbol{\sigma}^2\right] + c_3\left[\mathbf{s},\boldsymbol{\sigma}^3\right], \tag{3}$$

where c_1, c_2, c_3 are arbitrary scalar functions of the invariants $Tr(\boldsymbol{\sigma})$, $Tr(\boldsymbol{\sigma}^2)$, $Tr(\boldsymbol{\sigma}^3)$, $Tr(\mathbf{s}\boldsymbol{\sigma})$, $Tr(\mathbf{s}\boldsymbol{\sigma}^2)$, and $[\mathbf{A},\mathbf{B}] \equiv \mathbf{AB} - \mathbf{BA}$ is the Lie bracket (the commutator) of two tensors $\mathbf{A},\mathbf{B}$. Some objective rates lead to oscillatory stress components versus time in simple shearing when the shear is increasing in a monotonous way (e.g., for a constant imposed shear rate). Such oscillatory behavior is in contradiction with experimental observations.

The problem of choosing an appropriate form of the objective stress rate in finite strain plasticity model gained some new insight in recent literature, adopting the point of view of symmetries of the set of constitutive equations. Symmetry structure proved to be important not only from a theoretical and constitutive viewpoint, but also from the point of view of computations. Symmetry requirements form a cornerstone in continuum mechanics. One may classify such symmetries as material symmetries (acting in the reference configuration, they determine what is called the material symmetry group of a given material), and spatial symmetries, which have to be universally satisfied by any theory of finite deformation, and are usually referred to as material frame indifference. Following an early work by Adeleke (1980), Ericksen (2000) considered the idea of general point transformations of both the Lagrangian coordinates and of the fields, thereby extending the concept of material symmetries described by a mere change of the reference configuration. Those generalized coordinate transformations are invariance properties in the sense that they are compatible with the Galilean invariance of the strain energy density function, namely, the material frame indifference is preserved. Going one step further, one may suspect in addition to material frame indifference (a postulate valid independently of the physical problem) the existence of hidden symmetries in the space of control and internal variables. Lie groups of such symmetries, in principle, can be computed from the set of governing equations for a specific problem. The main originality advocated in the present contribution is the use of symmetry analysis as a methodology to provide the best possible description of the real behavior of materials.

The Lie symmetries of finite strain perfectly plastic equations were computed in Liu (2004) and numerical schemes preserving group properties were accordingly developed, having the advantage of satisfying the consistency condition exactly. Numerical schemes which preserve symmetry and utilize some induced conservation laws have long term stability and are endowed with improved efficiency and stability. In particular, the enforcement of the consistency condition at each time step usually requires some iterative method for the stress point to be mapped back to the yield surface (Simo and Hughes, 1998), and this is recognized as a main source of numerical errors.

The current paper is organized as follows. In Section 2, a constitutive model for finite strain rigid plasticity with kinematic hardening is discussed. It employs a second-order expansion of the plastic spin, and thus involves two constitutive functions. Subsequently, as a specific example, the case of simple shear is considered. For this case, a coupled system of two nonlinear ordinary differential equations (ODEs) is derived for back stress components; it involves one constitutive function.

In Section 3, a partial classification of the constitutive models is performed, in terms of the non-specific free functions, based on the analysis of symmetry properties and integrating factors yielding conserved quantities (constants of motion) admitted by the underlying ODEs. For models admitting extended symmetry properties and constants of motion, examples of exact and numerical solutions for back stress components are presented, and their physical applicability is discussed. It turns out that physical solutions can arise for a number of forms of constitutive functions.

A discussion of the obtained constitutive models is presented in Section 3.5. We end with a conclusion and perspectives in Section 4.

The notation used in the current paper is as follows. Vectors and second order tensors are denoted with boldface symbols. Lagrangian coordinates of material points of a solid body are denoted by the label $\mathbf{y}$; they are mapped to the Eulerian coordinates $\mathbf{x} = \mathbf{x}(\mathbf{y}, t)$. The transformation gradient (tangent mapping) is given by $\mathbf{F} = \nabla_{\mathbf{y}}\mathbf{x}$. The second order identity tensor is denoted by $\mathbf{I}$; $Tr(\cdot)$ is the trace operator; $\mathbf{A}^T$ is the transpose of $\mathbf{A}$; $\mathbf{A}^D = \mathbf{A} - \frac{1}{3}Tr(\mathbf{A})\mathbf{I}$ represents the deviatoric (traceless) part of $\mathbf{A}$.

2. Constitutive model for kinematic hardening

2.1. An extended formulation of Dafalias model

There exists vast literature devoted to the kinematics of finite strain elastoplasticity (e.g. Drucker, 1951). In the current work, we restrict to the rigid plastic case.

Dafalias (1985b) constructed a so-called constitutive spin for a rigid plastic material as the difference between the total spin and the plastic spin:

$$\boldsymbol{\omega} := \mathbf{W} - \mathbf{W}^p. \tag{4}$$

In Dafalias (1998), the following definition of the plastic spin was proposed for the investigation of the kinematic hardening:

$$\mathbf{W}^p = c(\mathbf{X}\mathbf{D}^p - \mathbf{D}^p\mathbf{X}) = c\left(\mathbf{X}^D\mathbf{D}^p - \mathbf{D}^p\mathbf{X}^D\right), \tag{5}$$

where c is a single material parameter, and $\mathbf{X}^D$ is the deviatoric part of $\mathbf{X}$. In the specific case of polycrystals, one has

$$\mathbf{W}^p = \langle\lambda\rangle\boldsymbol{\Omega}^p, \tag{6}$$

where $\boldsymbol{\Omega}^p$ represents the plastic rotation of the lattice. The writing of a specific constitutive relation for the plastic spin is supported by the idea that the rotation of the texture must be governed by a rule on its own, independently of the kinematics of deformation, see also Mandel (1971) and Kratochvil (1971).

According to the last equality in (5), for simplicity of notation, one may omit the superscript D for the deviatoric part of $\mathbf{X}$.

The next step is to prescribe a *kinematic hardening rule*, first written in the general form as

$$\widehat{\mathbf{X}} = F(\mathbf{X}, \boldsymbol{\sigma}; \mathbf{D}^p, \ldots), \tag{7}$$

where $\widehat{\mathbf{X}}$ represents an objective rate of the back stress $\mathbf{X}$. The objective stress rate is given, within the Dafalias model (Dafalias, 1998), by

$$\widehat{\mathbf{X}} = \dot{\mathbf{X}} + \mathbf{X}\boldsymbol{\omega} - \boldsymbol{\omega}\mathbf{X}, \tag{8}$$

and involves the expression of the constitutive spin itself based on the plastic spin in (5).

The expansion of a general form of the plastic spin (5) can be obtained using the representation theorem for an antisymmetrical anisotropic tensorial function (Boehler, 2006), as originally proposed in Dafalias (1985a,b), and will be written in a more open formulation. This is motivated by the fact that the objective rate of the stress deviator has to be a nonlinear expression of the back stress deviator. We now consider a second-order expansion of the plastic spin, given by

$$\mathbf{W}^p(\mathbf{X}, \mathbf{D}^p) = c_1(\mathbf{X}\mathbf{D}^p - \mathbf{D}^p\mathbf{X}) + c_2\left(\mathbf{X}^2\mathbf{D}^p - \mathbf{D}^p\mathbf{X}^2\right), \tag{9}$$

According to the representation theorems for antisymmetric isotropic functions (Boehler, 2006), the scalar coefficients c_1, c_2 may depend on invariants of the tensors $\mathbf{D}^p, \mathbf{X}$:

$$Tr(\mathbf{D}^p\mathbf{X}), \quad Tr\left(\mathbf{D}^p\mathbf{X}^2\right), \quad Tr\left((\mathbf{D}^p)^2\mathbf{X}\right), \quad Tr\left((\mathbf{D}^p)^2\mathbf{X}^2\right). \tag{10}$$

Coming back to the objective stress rate in (8), using (4) and (9), it can be expressed in terms of the Lie bracket as

$$\widehat{\mathbf{X}} = \dot{\mathbf{X}} + [\mathbf{X}, \mathbf{W}] - c_1[\mathbf{X}, [\mathbf{X}, \mathbf{D}^p]] - c_2\left[\mathbf{X}, \left[\mathbf{X}^2, \mathbf{D}^p\right]\right]. \tag{11}$$

It is quite complicated to evaluate the scalar functions of the invariants involved in such an expression (Khan and Huang, 1995), and the question of the right formulation of objective rates remains open. We presently advocate a novel contribution with respect to this issue, based on Lie symmetries as a tool to discriminate amongst all possible forms of the objective rates covered by the model (9), (11). Note the replacing the objective rate for the back stress in (11) by another stress measure, e.g., the Kirchhoff stress, leads to recovering Liu's approach (Liu, 2004) for the case of zero values of c_1, c_2. Classical objective rates can be recovered this way, such as the Jaumann rate, the Truesdell rate, or the Xiao-Bruhns-Meyers rate. One may therefore consider (11) with $c_1, c_2 \neq 0$ as an extension of Liu's model of the objective stress rate. In (11), the term $\dot{\mathbf{X}} + [\mathbf{X}, \mathbf{W}]$ linear in $\mathbf{X}$ is recognized as the Jaumann rate of the back stress, and the subsequent terms are quadratic in the back stress.

Since the kinematic hardening is related to the spin of the microstructure, it is quite natural to consider finite simple shear in the plastic regime as a test model. Simple shear is commonly used to study the properties of corotational rates (Khan and Huang, 1995). Below we restrict the general model (11) to Prager kinematic hardening.

2.2. The case of a simple shear

For simplicity of the formulation of constitutive equations, from now on, elastic deformations will be neglected, and a rigid plasticity model is considered. This is a reasonable assumption for the case of metals.

We consider the linear kinematic hardening rule of Prager's type, given by

$$\hat{\mathbf{X}} = \frac{2h}{3}\mathbf{D}^p, \tag{12}$$

which is a specific case of (7). In (12), $h = \text{const}$ is the hardening modulus, defined as the slope of the stress-plastic curve

$$\dot{\sigma}_e = hD_e^p,$$

where

$$D_e^p = \left(\frac{2}{3}\mathbf{D}^p : \mathbf{D}^p\right)^{1/2} = \left(\frac{2}{3}Tr(\mathbf{D}^p\mathbf{D}^p)\right)^{1/2} \tag{13}$$

is the equivalent plastic strain rate, and σ_e is the equivalent stress constructed from the Cauchy stress $\boldsymbol{\sigma}$.

The simple shear test is challenging due to the occurrence of a strong rotation (Duchene et al. (2008); Johansson (2008)).

Focusing on the case of the simple shear, the kinematics in the reference $(y_1, y_2)-$plane is obtained from the point transformation from the Lagrange to the Euler coordinates

$$x_1 = y_1 + \gamma y_2, \quad x_2 = y_2. \tag{14}$$

Since we presently consider an incompressible model, a straightforward calculation shows that the set of equations for the back stress components will remain the same in both cases of plane strain and plane stress. One consequently has

$$\mathbf{F} = \mathbf{I} + \gamma \mathbf{e}_1 \otimes \mathbf{e}_2. \tag{15}$$

Let the shear angle $\gamma(t)$ be given as an increasing function of time, $d\gamma/dt = 2\omega(t)$. It follows that the velocity gradient and its symmetrical and antisymmetrical parts are given by

$$\begin{aligned} &\mathbf{L} = 2\omega \mathbf{e}_1 \otimes \mathbf{e}_2, \\ &\mathbf{D}^p = \mathbf{D} := [\mathbf{L}] = \frac{1}{2}\left(\mathbf{L} + \mathbf{L}^T\right) = \omega(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1), \\ &\mathbf{W} = \mathbf{D} - [\mathbf{L}] = \omega(\mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1). \end{aligned} \tag{16}$$

The back stress is given by a symmetric matrix

$$\mathbf{X} = X_{11}\mathbf{e}_1 \otimes \mathbf{e}_1 + X_{12}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1) + X_{22}\mathbf{e}_2 \otimes \mathbf{e}_2. \tag{17}$$

A straightforward calculation using (9) leads to the expression for the constitutive spin which is given by

$$\boldsymbol{\omega} = \mathbf{W} - \mathbf{W}^p = f\mathbf{W}, \tag{18}$$

where

$$f = f(X_{11}, X_{12}, X_{22}) = 1 - c_1(X_{11} - X_{22}) - c_2\left((X_{11})^2 - (X_{22})^2\right), \tag{19}$$

and c_1, c_2 are, in general, functions of the scalar invariants (10). Dynamic ODEs for the three components of the back stress tensor are obtained from formulas (8), (12). In the matrix form, they are given by

$$\dot{\mathbf{X}} = \boldsymbol{\omega}\mathbf{X} - \mathbf{X}\boldsymbol{\omega} + \frac{2h}{3}\mathbf{D}^p, \tag{20}$$

and in the component form, by

$$\frac{dX_{11}}{d\gamma} - fX_{12} = 0, \quad \frac{dX_{22}}{d\gamma} + fX_{12} = 0, \quad \frac{dX_{12}}{d\gamma} + \frac{1}{2}f(X_{11} - X_{22}) = \frac{h}{3}. \tag{21}$$

From now on, we consider the natural case of zero initial conditions: $X_{11}(0) = X_{22}(0) = 0$. The first two equations of (21) yield $d/d\gamma(X_{11} + X_{22}) = 0$. Hence one has, for all times,

$$X_{11} + X_{22} \equiv 0. \tag{22}$$

It immediately follows that in (19), the term involving c_2 vanishes, and one has

$$f = f(X_{11}, X_{12}) = 1 - 2c_1X_{11}. \tag{23}$$

Finally, the components of the back stress are found by solving the equations

$$\begin{aligned} &\frac{dX_{11}}{d\gamma} - (1 - 2c_1X_{11})X_{12} = 0, \\ &\frac{dX_{12}}{d\gamma} + (1 - 2c_1X_{11})X_{11} = \frac{h}{3}, \\ &X_{22} = -X_{11}. \end{aligned} \tag{24}$$

Due to (24), the invariants (10) reduce to

$$\begin{aligned} &Tr(\mathbf{D}^p\mathbf{X}) = 2\omega X_{12} =: I_1, \quad Tr\left((\mathbf{D}^p)^2\mathbf{X}^2\right) = 2\omega^2(X_{11}^2 + X_{12}^2) =: I_2, \\ &Tr\left(\mathbf{D}^p\mathbf{X}^2\right) = Tr\left((\mathbf{D}^p)^2\mathbf{X}\right) = 0, \end{aligned} \tag{25}$$

and one has

$$c_1 = c_1(I_1, I_2). \tag{26}$$

A plastic flow condition of Von Mises type is introduced as

$$q(\boldsymbol{\sigma}, \mathbf{X}) = 0 \to Tr\left\{(\mathbf{S} - \mathbf{X})^2\right\} - \frac{2}{3}\sigma_Y^2 = 0, \tag{27}$$

where σ_Y is the initial yield strength. The stress is computed from the back stress and using the plastic flow rule and the given strain rate tensor $\mathbf{D}$ as

$$\mathbf{D} = \dot{\lambda}(\mathbf{S} - \mathbf{X}),$$

and hence one has

$$S_{11} = X_{11}, \quad S_{22} = -S_{11}, \quad S_{12} = X_{12} + \frac{\omega}{\dot{\lambda}}, \tag{28}$$

where $\dot{\lambda}$ the plastic multiplier, which can be determined versus the imposed strain rate $\dot{\gamma} = 2\omega$ using the plastic flow condition (27), and the normality condition:

$$\dot{\lambda} = \sqrt{3}\frac{\omega}{\sigma_Y}. \tag{29}$$

Note that $\dot{\lambda} \geqslant 0$, since D_{12} must be of the same sign as ω.

The Cauchy stress is finally obtained as

$$\boldsymbol{\sigma} = \frac{1}{3}Tr(\boldsymbol{\sigma}) + \mathbf{S}, \tag{30}$$

hence one has

$$\sigma_{11} = S_{22} + 2S_{11}, \quad \sigma_{22} = S_{11} + 2S_{22}, \quad \sigma_{12} = S_{12}, \tag{31}$$

Using (28) and (31), the Cauchy stress components are related to the back stress components by

$$\sigma_{11} = X_{11}, \quad \sigma_{22} = X_{22}, \quad \sigma_{12} = X_{12} + \frac{1}{\sqrt{3}}\sigma_Y. \tag{32}$$

Below we analyze the system of ODEs (24) for the back stress tensor components X_{11}, X_{12}, taking into account the fact that this system involves a priori arbitrary function $c_1(I_1, I_2)$.

2.3. A general autonomous ODE for simple shear

Using some algebra, the ODEs in the system (24) can be rewritten in the following compact way. We use the notation (23). Multiplying the first equation by $X_{11}(\gamma)$ and the second equation by $X_{12}(\gamma)$, and adding equations, one gets

$$\frac{1}{2}\frac{d}{d\gamma}\left(X_{11}^2+X_{12}^2\right)=\frac{1}{2}\frac{d}{d\gamma}I_1=\frac{h}{3}X_{12}. \tag{33}$$

Further, denoting

$$Z(\gamma)=\int_0^\gamma X_{12}(s)ds$$

and assuming initial conditions $X_{11}(0)=X_{12}(0)=0$ and $X_{11}(\gamma)\geqslant 0$, one has $Z(0)=0$, and

$$X_{11}=\sqrt{\frac{4}{3}h\omega Z-\left(\frac{dZ}{d\gamma}\right)^2},\quad X_{12}=\frac{dZ}{d\gamma}. \tag{34}$$

Denote the right-hand side of the second equation of (24) by

$$Q\left(Z,\frac{dZ}{d\gamma}\right):=\frac{h}{3}-f(X_{11},X_{12})\sqrt{\frac{2h}{3}Z-\left(\frac{dZ}{d\gamma}\right)^2}. \tag{35}$$

Then the second equation of (24) yields *a general second-order autonomous nonlinear ODE*

$$\frac{d^2Z}{d\gamma^2}=Q\left(Z,\frac{dZ}{d\gamma}\right), \tag{36}$$

which is equivalent to the system (24), provided that the form of the function Q is chosen in a way that corresponds to a physically relevant form of $f(X_{11},X_{12})$ in (24).

In particular, it follows that if $Z(\gamma)$ is a solution of the problem

$$\begin{aligned}&\frac{d^2Z}{d\gamma^2}=Q\left(Z,\frac{dZ}{d\gamma}\right),\\&Z(0)=Z'(0)=0\end{aligned} \tag{37}$$

for a given form of $Q\left(Z,\frac{dZ}{d\gamma}\right)$, then (34) is a solution of the ODE problem

$$\begin{aligned}&\frac{dX_{11}}{d\gamma}-fX_{12}=0,\\&\frac{dX_{12}}{d\gamma}+fX_{11}=\frac{h}{3},\\&X_{11}(0)=X_{12}(0)=0\end{aligned} \tag{38}$$

with the corresponding f found from (35), and the related constitutive coefficient c_1 determined by (23).

Remark 1. Though simple-looking, the ODE problem (37) is perhaps less practically useful for classifications below than the original system (38), due to the technical difficulty of tracking physical feasibility of forms of the function Q compared to $f(X_{11},X_{12})$. In particular, one needs to choose forms of Q corresponding to real-valued X_{11} in (34).

Remark 2. As it is well-known, since the ODE (36) is invariant with respect to translations $\gamma\to\gamma+\text{const}$, therefore one can use canonical variables

$$Z(\gamma)=Z,\quad \frac{dZ}{d\gamma}=N(Z),\quad \frac{d^2Z}{d\gamma^2}=N(Z)\frac{dN(Z)}{dZ},$$

to further reduce the ODE (36) to a first-order equation

$$\frac{dN(Z)}{dZ}=\frac{1}{N(Z)}Q(Z,N(Z)). \tag{39}$$

Remark 3. Another possible representation of the system (24) in the form of a single ODE is the straightforward phase-plane representation

$$\frac{dX_{12}}{dX_{11}}=-\frac{X_{11}}{X_{12}}+\frac{h}{3X_{12}f(X_{11},X_{12})}. \tag{40}$$

3. Symmetries and Integrating Factors for the Case of Simple Shear

Continuous symmetries have proven to be useful in applied mathematics and mechanics (e.g. Olver, 2000; Ibragimov, 2002; Bluman et al., 2010). The importance of the continuous symmetries inherent to the constitutive mechanical laws of dissipative materials has been underlined in many contributions since the foundations of continuum mechanics. Symmetries play a major role in the construction of the constitutive laws of materials, given from a fundamental viewpoint by tensorial functionals that have to be objective (traducing the invariance of the material's response under the group of rotations in the three-dimensional Euclidean space) and invariant under the action of the material symmetry group (a group of discrete symmetries) of the material (Weyl, 1997; Fulton and Harris, 1991). The thermodynamics of irreversible processes is the natural framework for handling the dissipative behavior of materials, involving the use of thermodynamic potentials (basically the free energy and a dissipation potential); those potentials should reflect the symmetry properties of the material or structure being analyzed. Considering for instance a general internal variable formulation of inelasticity for generalized standard materials, some of those symmetries are implicitly reflected in the form of the thermodynamic potential and the dissipation function. This thermodynamical framework for the writing of the constitutive laws of dissipative materials has its roots in the works by Biot (1965), Ziegler (1963), Germain (1973) and Halphen et al. (1975), and has proven its ability to cover a broad spectrum of models in viscoelasticity, viscoplasticity, plasticity and also continuum damage mechanics. The ideal plasticity equations were analyzed within this spirit in Senashov (1980), Annin et al. (1985), Mielke, 2002 related finite elastoplasticity to Lie groups and geodesics. Recently, Rajagopal and Srinivasa (1998) extended the classical notion of material symmetry, within the framework of multiple natural configurations, accounting for a change of the material symmetry group due to microstructural changes.

3.1. Computation of point symmetries

Symmetries of a system of differential equations (DEs) are defined as transformations that leave invariant the solution manifold of the system, i.e, any solution of the system is mapped into a solution of the same system. One can use Lie's algorithm to systematically find Lie groups of point (more generally, local) symmetries of differential equations (see, e.g. Olver, 2000; Bluman et al., 2010).

Consider a DE system $\mathbf{E}\{\mathbf{x},\mathbf{u}(\mathbf{x})\}=0$, where $\mathbf{E}\{\mathbf{x},\mathbf{u}(\mathbf{x})\}=(E^1\{\mathbf{x},\mathbf{u}(\mathbf{x})\},\ldots,E^N\{\mathbf{x},\mathbf{u}(\mathbf{x})\})$ are N expressions involving independent variables $\mathbf{x}=(x^1,\ldots,x^n)$ and dependent variables $\mathbf{u}(\mathbf{x})=(u^1(\mathbf{x}),\ldots,u^m(\mathbf{x}))$ and derivatives of $\mathbf{u}(\mathbf{x})$ up to some given order. For such a system, a Lie group of point symmetries has the form

$$\begin{aligned}(x^*)^i &= f^i(x,u;\varepsilon)=x^i+\varepsilon\xi^i(x,u)+O(\varepsilon^2),\quad i=1,\ldots,n,\\ (u^*)^\mu &= g^\mu(x,u;\varepsilon)=u^\mu+\varepsilon\eta^\mu(x,u)+O(\varepsilon^2),\quad \mu=1,\ldots,m,\end{aligned} \tag{41}$$

where ε is a group parameter. The transformations (41) are symmetries of a given DE system if

$$\mathbf{E}\{\mathbf{x}^*,\mathbf{u}^*(\mathbf{x}^*)\}=0\quad\text{when}\quad \mathbf{E}\{\mathbf{x},\mathbf{u}(\mathbf{x})\}=0.$$

The knowledge of infinitesimal components $\{\xi^i,\eta^\mu\}$ is equivalent to the knowledge of the global group action $\{f^i,g^\mu\}$ by the first Lie's theorem. Symmetries are found from solving linear *determining equations* for the unknown symmetry components $\{\xi^i,\eta^\mu\}$. Instead of the global form in (41), symmetries are often given in terms of infinitesimal generators

$$\mathrm{Y}=\xi^i\frac{\partial}{\partial x^i}+\eta^\mu\frac{\partial}{\partial u^\mu}, \tag{42}$$

where summation in repeated indices is assumed.

In the current paper, we concentrate our attention on the ODE system in the case of simple shear

$$\begin{aligned}&\frac{dX_{11}}{d\gamma}-fX_{12}=0,\\ &\frac{dX_{12}}{d\gamma}+fX_{11}=\frac{h}{3},\end{aligned} \tag{43}$$

given by the first two equations of (24), with the only independent variable γ, and two dependent variables $X_{11}(\gamma),X_{12}(\gamma)$; hence its point symmetries are sought in the form

$$\mathrm{Y}=\xi\frac{\partial}{\partial\gamma}+\eta^1\frac{\partial}{\partial X_{11}}+\eta^2\frac{\partial}{\partial X_{12}}, \tag{44}$$

where the symmetry components ξ,η^1,η^2 depend, in general, on γ,X_{11},X_{12}. An important feature of the system (43) is that it involves an arbitrary function $f(X_{11},X_{12})$ that defines the spin matrix (18), and hence the objective stress rate through (7). Below, we classify point symmetries of the system (43) to isolate specific forms of $f(X_{11},X_{12})$ that lead to extended symmetry structure.

3.2. Computation of integrating factors

Again, for generality, consider a system $\mathbf{E}\{\mathbf{x},\mathbf{u}(\mathbf{x})\}=0$ of N differential equations with independent variables $\mathbf{x}=(x^1,\ldots,x^n)$ and dependent variables $\mathbf{u}(\mathbf{x})=(u^1(\mathbf{x}),\ldots,u^m(\mathbf{x}))$. *Conservation laws* given by nontrivial divergence expressions

$$\frac{\partial\Psi^i\{\mathbf{x},\mathbf{u}(\mathbf{x})\}}{\partial x^i}=0 \tag{45}$$

that vanish on solutions of a given system $\mathbf{E}\{\mathbf{x},\mathbf{u}(\mathbf{x})\}=0$ can be found systematically using the direct construction method (Anco and Bluman, 2002; Bluman et al., 2010). The method proceeds by finding N conservation law multipliers $\Lambda_q\{\mathbf{x},\mathbf{u}(\mathbf{x})\}, q=1,\ldots,N$, such that a linear combination of given equations with these multipliers yields a conservation law:

$$\Lambda_q\{\mathbf{x},\mathbf{u}(\mathbf{x})\}E^q\{\mathbf{x},\mathbf{u}(\mathbf{x})\}\equiv\frac{\partial\Psi^i\{\mathbf{x},\mathbf{u}(\mathbf{x})\}}{\partial x^i}=0. \tag{46}$$

The multipliers $\Lambda_q\{\mathbf{x},\mathbf{u}(\mathbf{x})\}$ are found from determining equations involving Euler differential operators (Anco and Bluman, 2002).

For ODEs with a single independent variable γ, a conservation law expression (46) becomes simply an expression

$$\Lambda_q\{\gamma,\mathbf{u}(\gamma)\}E^q\{\gamma,\mathbf{u}(\gamma)\}\equiv\frac{\partial\Psi\{\gamma,\mathbf{u}(\gamma)\}}{\partial\gamma}=0 \tag{47}$$

relating *integrating factors* $\Lambda_q\{\gamma,\mathbf{u}(\gamma)\}$ and a *first integral* (constant of motion) $\Psi\{\gamma,\mathbf{u}(\gamma)\}$. Determining equations for admitted sets of integrating factors are obtained by, first, specifying an ansatz (dependence of the integrating factors), and second, applying Euler operators with respect to each dependent variable u^j:

$$\mathcal{E}_{u^j}\left(\Lambda_q\{\gamma,\mathbf{u}(\gamma)\}E^q\{\gamma,\mathbf{u}(\gamma)\}\right)=0,\quad j=1,\ldots,m. \tag{48}$$

Eqs. (48) are subsequently solved off of the solution space of the given system $\mathbf{E}\{\gamma,\mathbf{u}(\gamma)\}=0$. First integral are subsequently computed through (47). For the ODE system (43) describing the simple shear, we denote its two equations by $E^1=E^1\{\gamma,X_{11}(\gamma),X_{12}(\gamma)\}=0$ and $E^2\{\gamma,X_{11}(\gamma),X_{12}(\gamma)\}=0$, and obtain two determining equations

$$\begin{aligned}&\mathcal{E}_{X_{11}}\left(\Lambda_1E^1+\Lambda_2E^2\right)=0,\\&\mathcal{E}_{X_{12}}\left(\Lambda_1E^1+\Lambda_2E^2\right)=0,\end{aligned} \tag{49}$$

for the unknown integrating factors $\Lambda_1=\Lambda_1\{\gamma,X_{11}(\gamma),X_{12}(\gamma)\}$, $\Lambda_2=\Lambda_2\{\gamma,X_{11}(\gamma),X_{12}(\gamma)\}$. In (49), the Euler operators are given by

$$\begin{aligned}\mathcal{E}_{X_{11}}&=\frac{\partial}{\partial X_{11}}-\mathrm{D}_\gamma\frac{\partial}{\partial\dot{X}_{11}}+\mathrm{D}_\gamma^2\frac{\partial}{\partial\ddot{X}_{11}}+\cdots,\\\mathcal{E}_{X_{12}}&=\frac{\partial}{\partial X_{12}}-\mathrm{D}_\gamma\frac{\partial}{\partial\dot{X}_{12}}+\mathrm{D}_\gamma^2\frac{\partial}{\partial\ddot{X}_{12}}+\cdots,\end{aligned}$$

where dots denote time derivatives, and the total time derivative is given by

$$\mathrm{D}_\gamma=\frac{\partial}{\partial\gamma}+\dot{X}_{11}\frac{\partial}{\partial X_{11}}+\dot{X}_{12}\frac{\partial}{\partial X_{12}}+\ddot{X}_{11}\frac{\partial}{\partial\dot{X}_{11}}+\ddot{X}_{12}\frac{\partial}{\partial\dot{X}_{12}}+\cdots.$$

Below, we solve determining Eqs. (49) for some specific forms of multipliers, and isolate cases of the arbitrary function $f(X_{11},X_{12})$ which correspond to the appearance of additional first integrals within the chosen forms of multipliers.

The obtained first integrals represent invariance relations between variables and parameters of the present finite strain constitutive model. Those invariants are useful to condense the response of inelastic materials and to predict changes of behaviors when some loading or material parameters do change. In an empirically-based work, Ferry, 1980 developed the so-called WLF time temperature equivalence principle. Recent works in this line of thoughts evidence the interest of the Lie symmetries as a predictive method to obtain invariance properties for materials with nonlinear viscoelastic behavior (Magnenet et al., 2003), which can be written under the umbrella of a thermodynamical framework of relaxation (Magnenet et al., 2007). The shift factor allowing to map the responses obtained for a continuous range of variation of the parameters into a reference curve (the so-called master curve) is there directly calculated in terms of the constitutive model parameters, thus allowing constructing the master curve. This master curve is nothing but the graphical expression of the invariance relations obtained from the group analysis.

3.3. Some classes of constitutive functions that yield symmetries and integrating factors for the case of simple shear

Since Eqs. (43) are both first-order equations, it follows that for each fixed form of $f(X_{11},X_{12})$, these equations have an infinite number of point symmetries and an infinite number of zeroth-order integrating factors. The problem of computation of point symmetries and zeroth-order integrating factors is often at least of the same complexity as the solution of Eqs. (43) themselves. Therefore a complete classification is not feasible.

It is straightforward, however, to determine point symmetries and integrating factors that hold for *all* forms of $f(X_{11}, X_{12})$. Here one finds only one point symmetry (time translation) given by

$$\mathrm{Y} = \frac{\partial}{\partial \gamma},$$

and no zeroth-order integrating factors.

We now perform a partial classification of point symmetries and zeroth-order integrating factors of equations (43) with respect to forms of $f(X_{11}, X_{12})$. We use specific ansätze for symmetry components and integrating factors, and aim at finding forms of $f(X_{11}, X_{12})$ that lead to richer symmetry/first integral structure within the chosen ansätze. Several cases of $f(X_{11}, X_{12})$ for which additional point symmetries and/or first integrals are found are presented below.

Experimental results indicate the following main properties that the response of the material should satisfy in terms of the evolution of the stress components versus shear strain, under an imposed monotonous shear.

(a) An oscillatory behavior should be avoided for all stress components.
(b) The shear stress should have the same sign as the shear strain.

This argumentation provides a guideline for the selection of the stress evolutions having a physical meaning. It accordingly filters the constitutive models obtained in the classification.

Case 1

$$f(X_{11}, X_{12}) = \frac{K}{X_{11}}, \quad K = \text{const}. \tag{50}$$

Here one has a first integral

$$J_{1,1} = X_{12} - \left(\frac{h}{3} - K\right)\gamma = C_1,$$

$$J_{1,2} = K\left(\frac{h}{3} - K\right)\gamma^2 - 2K\gamma X_{12} + X_{11}^2 = C_2,$$

$$C_1, \quad C_2 = \text{const},$$

and the general solution reads

$$X_{11} = \sqrt{K\left[\left(\frac{h}{3} - K\right)\gamma^2 + C_2\gamma\right] + C_1}, \quad X_{12} = \left(\frac{h}{3} - K\right)\gamma + C_2. \tag{51}$$

This corresponds to a linear form of the shear stress component X_{12}, and an asymptotically linear X_{11}, and requires $K < h/3$ for the global existence of solution for $\gamma > 0$.

In the case of zero initial conditions $X_{11}(0) = X_{12}(0) = 0$, the solution (51) formally exists, and is given by linear functions of time:

$$X_{11} = \gamma\sqrt{K\left(\frac{h}{3} - K\right)}, \quad X_{12} = \left(\frac{h}{3} - K\right)\gamma. \tag{52}$$

Case 2

$$f(X_{11}, X_{12}) = \frac{\sqrt{AX_{11}^2 + B}}{X_{11}}, \quad C_1, C_2 = \text{const}. \tag{53}$$

For this case, Eqs. (43) can be shown to admit a first integral given by

$$J_{2,1} = 3C_1(X_{11}^2 + X_{12}^2) - 2h\sqrt{C_1 X_{11}^2 + C_2}.$$

Through the consideration of integrating factors depending on γ, X_{11}, one finds two additional first integrals, which are better presented in specific subcases.

Subcase 2a:

$$f(X_{11}, X_{12}) = \pm\frac{\sqrt{N^2 + M^2 X_{11}^2}}{X_{11}}, \quad M, N = \text{const}. \tag{54}$$

Here the two additional invariants are given by

$$J_{2,2} = 3MX_{12}\sin(M\gamma) + (h - 3\sqrt{M^2X_{11}^2 + N^2})\cos(M\gamma),$$

$$J_{2,3} = 3MX_{12}\cos(M\gamma) - (h - 3\sqrt{M^2X_{11}^2 + N^2})\sin(M\gamma).$$

Using the constants of motion $J_{2,2}, J_{2,3}$, one readily computes the general solution for the back stress components. They are given by

$$X_{11} = \frac{1}{3M}\sqrt{(h - 2Q\cos(M\gamma + \phi))^2 - 9N^2}, \quad X_{12} = \frac{Q}{3M}\sin(M\gamma + \phi), \tag{55}$$

where Q, ϕ are constants of integration given in terms of the invariants by

$$J_{2,2} = Q\cos\phi, \quad J_{2,3} = Q\sin\phi.$$

The back stress components (55) are oscillatory and thus do not correspond to a physically desirable objective rate.

Subcase 2b:

$$f(X_{11}, X_{12}) = \pm\frac{\sqrt{N^2 - M^2X_{11}^2}}{X_{11}}, \quad M, N = \text{const}. \tag{56}$$

Here the two additional invariants are given by

$$J_{2,2} = e^{M\gamma}(3\sqrt{N^2 - M^2X_{11}^2} + 3MX_{12} - h),$$

$$J_{2,3} = e^{-M\gamma}(3\sqrt{N^2 - M^2X_{11}^2} - 3MX_{12} - h).$$

For the case of zero initial conditions for the back stress components, the exact solution is given by

$$\begin{aligned} X_{11} &= \frac{1}{6M}\sqrt{(h - 3N)[8h\cosh(M\gamma) - 6(N + h)] - 2(h - 3N)^2\cosh(2M\gamma)}, \\ X_{12} &= \frac{1}{3M}(h - 3N)\sinh(M\gamma). \end{aligned} \tag{57}$$

Due to the exponential growth of the $\cosh(2M\gamma)$ term, the component X_{11} evidently fails to exist for all shear values. An exception is the case when $h = 3N$, which yields a trivial solution.

Case 3

$$f(X_{11}, X_{12}) = \frac{Q_3(X_{12})}{X_{11}}, \tag{58}$$

where $Q_3(X_{12})$ is an arbitrary function. For this case, one has a first integral given by

$$J_{3,1} = M_3(X_{12}) + \frac{\gamma}{3},$$

where the function $M_3(X_{12})$ is related to $Q_3(X_{12})$ by

$$Q_3(X_{12}) = \frac{1}{3(M_3'(X_{12}) + h)}.$$

Moreover, for Case 3, there exists an infinite number of point symmetries in a special ansatz where the dependence of symmetry components ξ, η^1, η^2 is restricted to γ, X_{12}.

Subcase 3a: In the subcase when

$$f(X_{11}, X_{12}) = \frac{1}{X_{11}(AX_{12} + B)}, \quad A, B = \text{const}, \tag{59}$$

the back stress Eqs. (43) admits an additional symmetry in an ansatz where ξ, η^1, η^2 depend on γ and X_{11} only, and moreover, an additional conserved quantity. The two conserved quantities are given by

$$\begin{aligned} J_{3a,1} &= 3AX_{11}^2 - 6X_{12}B + 2(Bh - 3)\gamma, \\ J_{3a,2} &= A^2hX_{11}^2 - 2Ah\gamma + 6B\ln[AhX_{12} + Bh - 3]. \end{aligned}$$

Consequently, the solution X_{11}, X_{12} satisfying $X_{11}(0) = X_{12}(0) = 0$ is given implicitly by

$$\begin{aligned} &3AX_{11}^2 - 6X_{12}B + 2(Bh-3)\gamma = 0, \\ &A^2hX_{11}^2 - 2Ah\gamma + 6B\ln[AhX_{12} + Bh - 3] = 6B\ln(Bh-3). \end{aligned} \tag{60}$$

Sample plots of solutions (60) satisfying physical requirements are given in Fig. 1.

Case 4

$$f(X_{11}, X_{12}) = \frac{Q_4(X_{11})}{X_{12}}, \tag{61}$$

where $Q_4(X_{11})$ is an arbitrary function. For this case, from the first ODE of (43), one obtains a first integral given by

$$J_{4,1} = M_4(X_{11}) - \gamma,$$

where $Q_4(X_{11}) = 1/M_4'(X_{11})$. To obtain a sample solution, consider

$$f(X_{11}, X_{12}) = \frac{\sqrt{N^2 + M^2X_{11}^2}}{X_{12}}, \quad M, N = \text{const}. \tag{62}$$

Here one has

$$J_{4,1} = M\gamma - \ln(MX_{11} + \sqrt{M^2X_{11}^2 + N^2}),$$

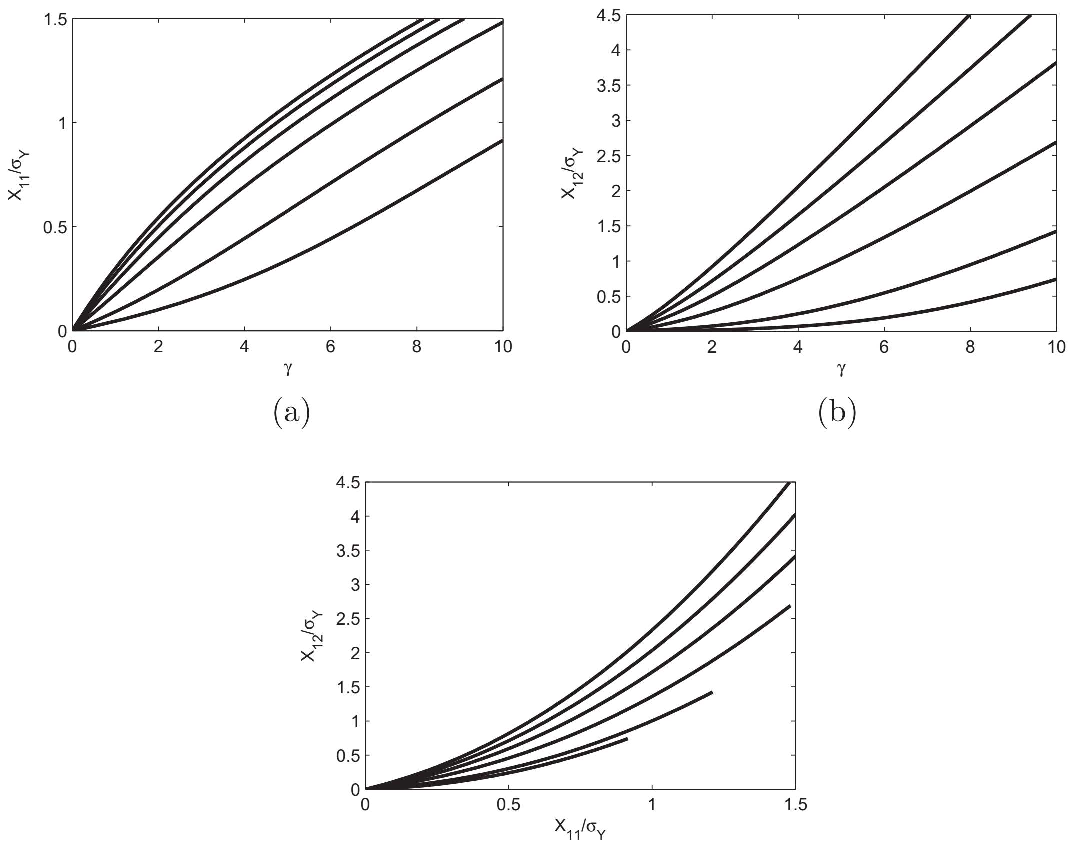

Fig. 1. Plots of back stress components X_{11}/σ_Y (a) and X_{12}/σ_Y (b) as functions of γ, and a phase diagram (c), for the case (59). The parameters used in plots are: $A = 5/\sigma_Y^2, B = 3.05/\sigma_Y, \gamma \in [0, 10]$, $h/\sigma_Y = 1, 1.05, 1.25, 1.5, 1.75, 2$ (bottom to top in each plot). (The results are independent of σ_Y. Typically, $\sigma_Y \sim 10^6$ Pa).

which leads to an expression for X_{11}. The solution satisfying $X_{11}(0) = 0$ is given by

$$X_{11} = \frac{N}{M}\sinh(M\gamma), \tag{63}$$

and X_{12} is the solution of the problem

$$\frac{dX_{12}}{d\gamma} = \frac{h}{3} + \frac{N^2}{2MX_{12}}\sinh(2M\gamma), \quad X_{12}(0) = 0. \tag{64}$$

Solutions (63), (64) exhibit exponential growth; sample plots are shown in Fig. 2.

Case 5

$$f(X_{11}, X_{12}) = Q_5(X_{11}), \tag{65}$$

where $Q_5(X_{11})$ is an arbitrary function. For this case, for any form of the arbitrary function $Q_5(X_{11})$, one has a first integral given by

$$J_{5,1} = \frac{X_{11}^2}{2} + \frac{X_{12}^2}{2} - \frac{h}{3}\int \frac{1}{Q_5(X_{11})}dX_{11}.$$

As a first example, in Fig. 3, we produce plots for

$$f(X_{11}, X_{12}) = \frac{Me^{-NX_{11}}}{X_{11} + A}, \quad A, M, N = \text{const}. \tag{66}$$

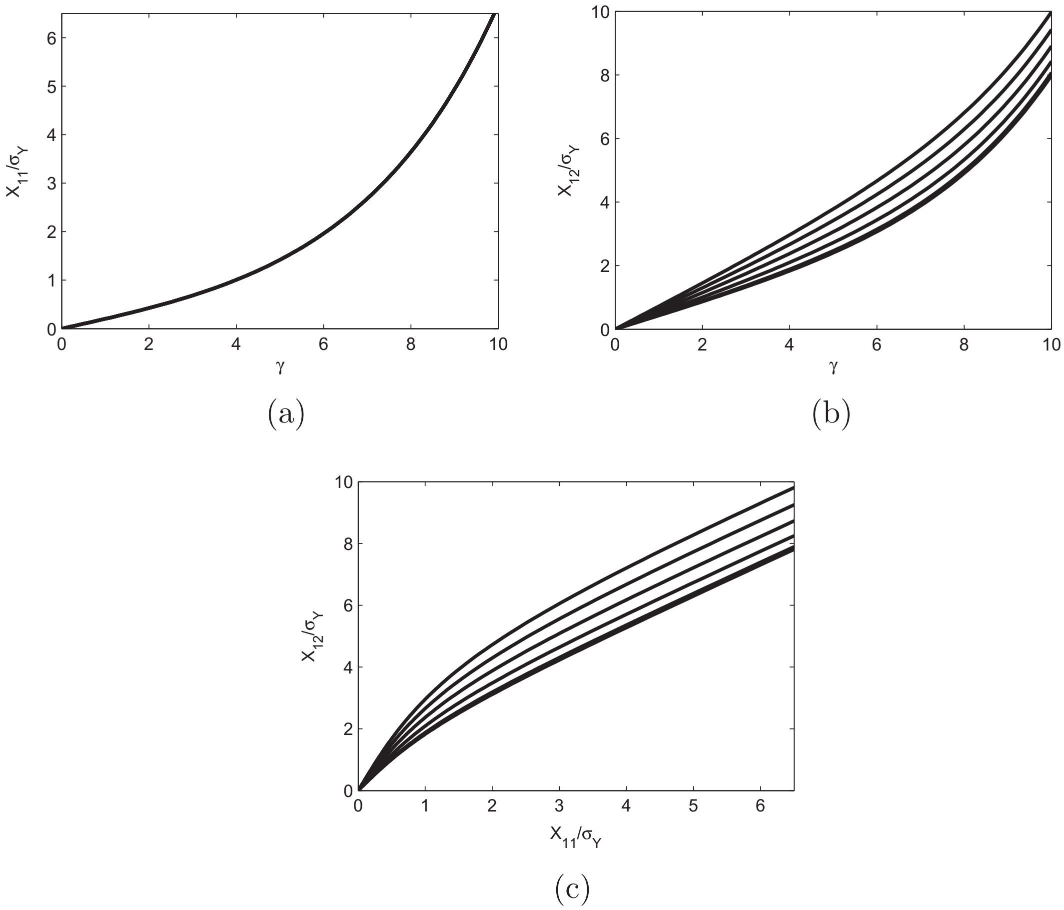

Fig. 2. Plots of back stress components X_{11}/σ_Y (a) and X_{12}/σ_Y (b) as functions of γ, and a phase diagram (c), for the case (62). The parameters used in plots are: $M = 0.3, N/\sigma_Y = 0.2, \gamma \in [0, 10],\ h/\sigma_Y = 1, 1.05, 1.25, 1.5, 1.75, 2$ (bottom to top). [The results are independent of σ_Y. Typically, $\sigma_Y \sim 10^6$ Pa].

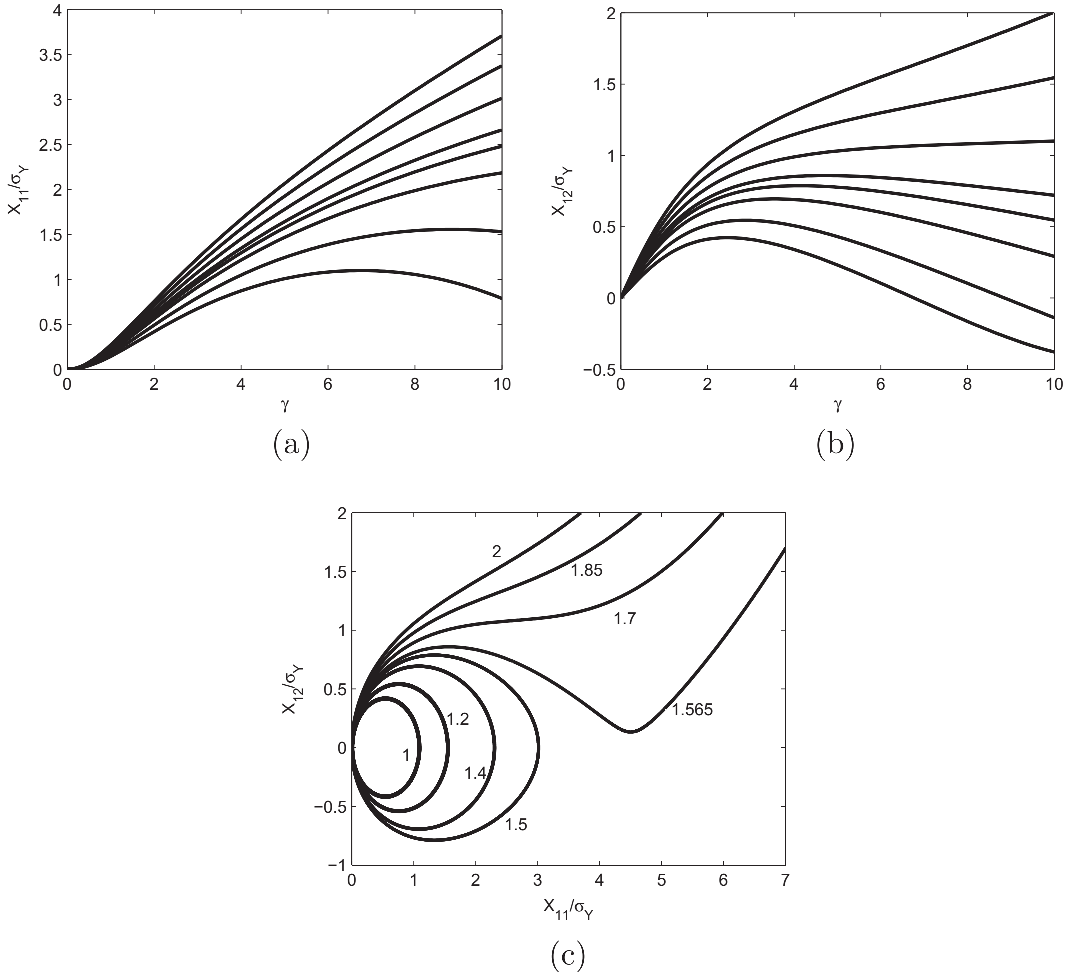

Fig. 3. Back stress components X_{11}/σ_Y (a) and X_{12}/σ_Y (b) as functions of γ, and a phase diagram (c), for the case (66). The parameters used in plots are: $A = M = \sigma_Y, N = 0.1/\sigma_Y, h/\sigma_Y = 1, 1.2, 1.4, 1.5, 1.565, 1.7, 1.85, 2$ (bottom to top in plots (a) and (b)).

For unstable states, the range of the ratio h/σ_Y has to be bounded below in order to maintain nonnegative values of the back stress X_{12}, as it is seen in Fig. 3(c).

As a second example, we consider forms of the constitutive function (65) which lead to the asymptotic behaviour

$$\lim_{\gamma\to\infty} \frac{dX_{12}}{d\gamma} = 0. \tag{67}$$

From the second equation of (43), one has

$$\frac{dX_{12}}{d\gamma} = \frac{h}{3} - X_{11}Q_5(X_{11}) \geqslant 0; \quad \lim_{\gamma\to\infty}\left(\frac{h}{3} - X_{11}Q_5(X_{11})\right) = 0.$$

This is evidently satisfied when

$$Q_5(X_{11}) = \frac{1}{X_{11}}\left(\frac{h}{3} - q(X_{11})\right)$$

with

$$0 \leqslant q(X_{11}) \leqslant \frac{h}{3}, \quad \lim_{\gamma\to\infty} q(X_{11}) = 0.$$

Taking, for example,

$$f(X_{11}, X_{12}) = Q_5(X_{11}) = \frac{1}{X_{11}}\left(\frac{h}{3} - \frac{A}{3}\exp(-M(X_{11})^N)\right), \quad A, M, N = \text{const}, \tag{68}$$

one can obtain solutions satisfying the condition (67), with behaviour shown in Fig. 4.

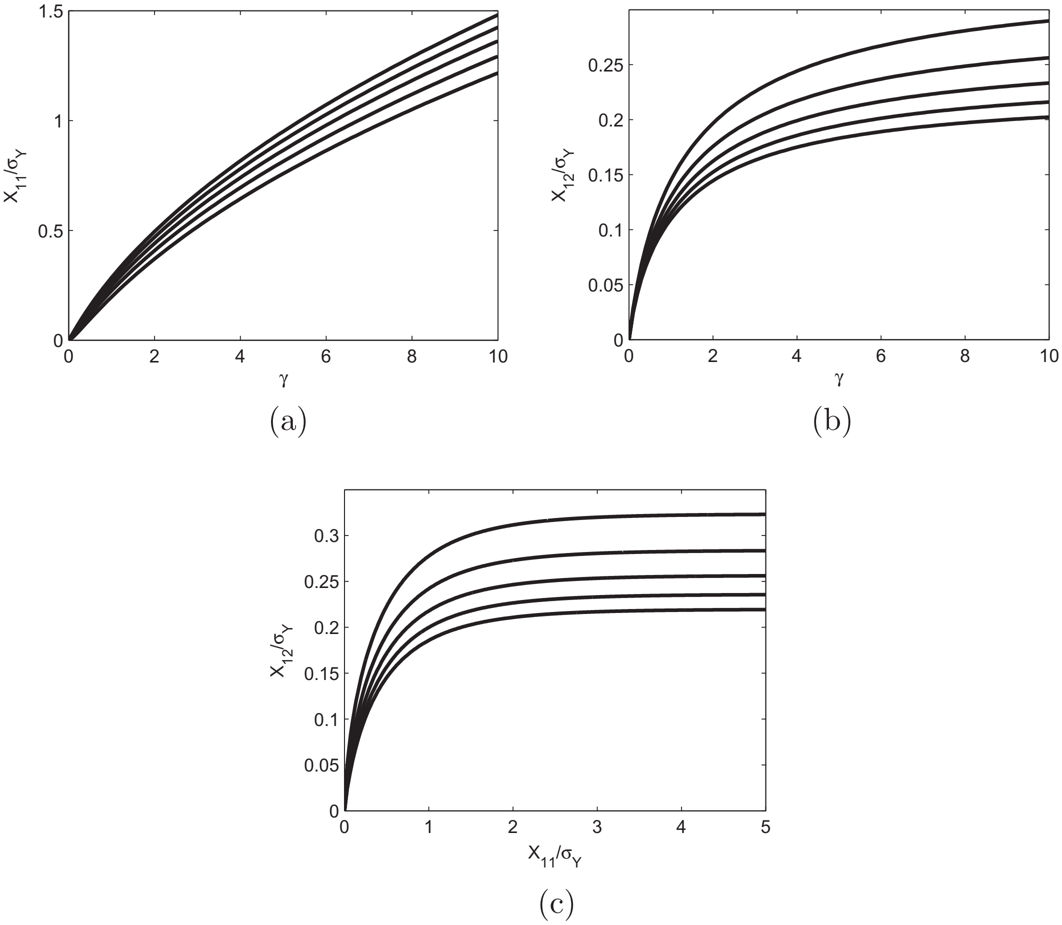

Fig. 4. Back stress components X_{11}/σ_Y (a) and X_{12}/σ_Y (b) as functions of γ, and a phase diagram (c), for the case (68). The parameters used in plots are: $A = \sigma_Y, N = 0.6, M = 4/(\sigma_Y)^N, h/\sigma_Y = 1, 1.25, 1.5, 1.75, 2$ (bottom to top in (a), top to bottom in (b) and (c)).

Case 6

$$f(X_{11}, X_{12}) = \frac{Q_6(X_{12})}{X_{11} + A}, \tag{69}$$

where $Q_6(X_{11})$ is an arbitrary function, and $A = \text{const} \neq 0$. In the case (69), it is straightforward to compute a first integral in the special case

$$Q_6(X_{12}) = M + \frac{N}{X_{12}}; \tag{70}$$

the first integral is given by

$$J_{6,1} = 3M(X_{11}^2 + X_{12}^2) - hX_{11}^2 + 2h(\gamma N - AX_{11}).$$

For the form (69), (70) of the constitutive function, depending on parameters A, M, N, one can have various types of behaviour of the stress components. For example, for the parameters used in Fig. 5, one has increasing stress components with an ultimate behaviour

$$\lim_{\gamma\to\infty} \frac{dX_{11}}{d\gamma} = \text{const}, \quad \lim_{\gamma\to\infty} \frac{dX_{12}}{d\gamma} = \text{const}.$$

Using different sets of parameters in the constitutive function (69), (70), an unstable behaviour according to Drucker's postulate ($\dot{X}_{12} < 0$, see Section 3.5) can be obtained for lower values of h. Sample curves for back stress components are given in Fig. 6.

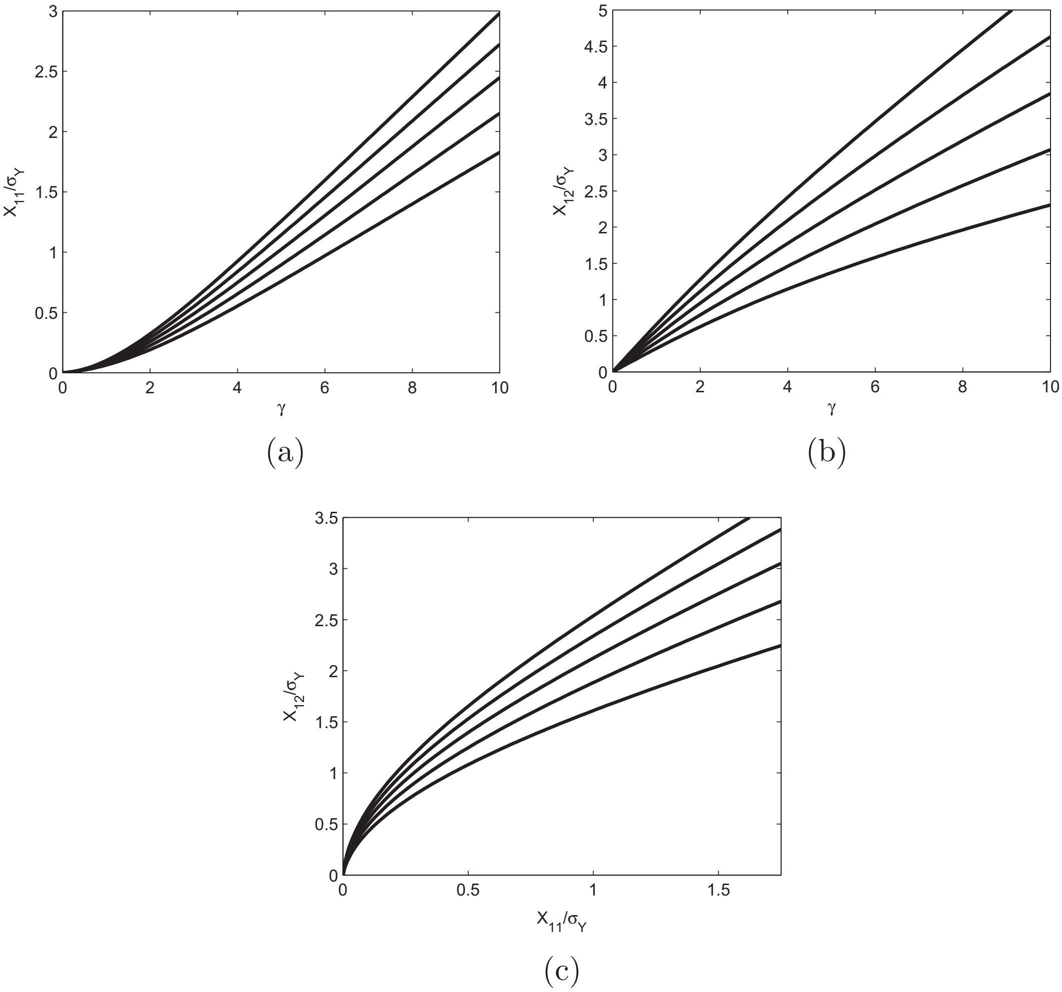

Fig. 5. Back stress components X_{11}/σ_Y (a) and X_{12}/σ_Y (b) as functions of γ, and a phase diagram (c), for the case (69), (70), for parameter values $A = \sigma_Y, M = 0.25\sigma_Y, N = 0.025\sigma_Y^2, h/\sigma_Y = 1, 1.25, 1.5, 1.75, 2$ (bottom to top in all plots).

3.4. Cases admitting particular classes of symmetries

We now seek forms of the constitutive function $f(X_{11}, X_{12})$ in the ODE system (38) for which the equations admit particular classes of point symmetries.

A. Translation symmetries.

- $Y = \frac{\partial}{\partial \gamma}$ admitted for all $f(X_{11}, X_{12})$.
- $Y = A\frac{\partial}{\partial X_{11}} + B\frac{\partial}{\partial X_{12}}, A^2 + B^2 > 0$, admitted only for $f(X_{11}, X_{12}) = 0$.

B. Rotational- or boost-type symmetries. Such symmetries do not arise for any form of $f(X_{11}, X_{12})$.

C. Scaling symmetries. A general scaling symmetry

$$Y = P\gamma\frac{\partial}{\partial \gamma} + QX_{11}\frac{\partial}{\partial X_{11}} + RX_{12}\frac{\partial}{\partial X_{12}}, \quad P^2 + Q^2 + R^2 > 0,$$

is admitted only in the case $P = Q = R$, when

$$f(X_{11}, X_{12}) = \frac{1}{X_{11}} S\left(\frac{X_{11}}{X_{12}}\right), \tag{71}$$

where $S(z)$ is an arbitrary function. For a special case when

$$S(z) = \frac{z}{Az + B}, \quad f(X_{11}, X_{12}) = \frac{1}{AX_{11} + BX_{12}}, \quad A, B = \text{const},$$

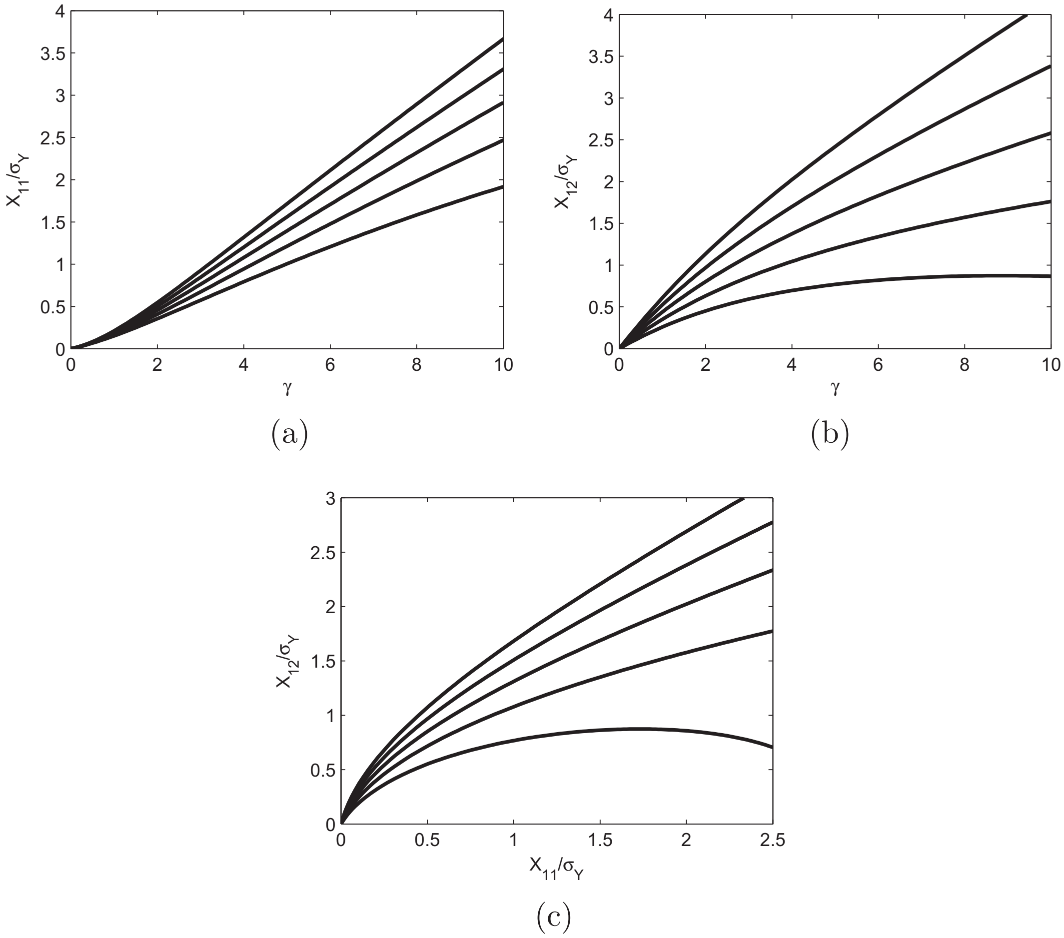

Fig. 6. Back stress components X_{11}/σ_Y (a) and X_{12}/σ_Y (b) as functions of γ, and a phase diagram (c), for the case (69), (70), for parameter values $A = \sigma_Y, M = 0.4\sigma_Y, N = 0.11\sigma_Y^2, h/\sigma_Y = 1, 1.25, 1.5, 1.75, 2$ (bottom to top in all plots).

a simple constant of motion can be obtained:

$$J = 3(AX_{12} - BX_{11}) + (3 - hA)\gamma.$$

In this case, however, the initial value problem (38) is degenerate, and admits an infinite number of linear solutions

$$X_{11} = \alpha\gamma, \quad X_{12}(t) = \beta\gamma, \quad \alpha = \frac{1}{B}\left(A\beta + 1 - \frac{hA}{3}\right), \quad \beta = \text{const}.$$

D. Projective-type symmetries. The ODE system (38) admits projective-type symmetries in a number of cases, including the following two cases.

Case (D1):

$$f(X_{11}, X_{12}) = \frac{X_{11}}{A(X_{11}^2 + X_{12}^2) - \frac{1}{2}BX_{12}^2 + CX_{11}^{-2B/A}}, \quad A, B, C = \text{const}. \tag{72}$$

In this case, the system (38) admits a projective symmetry

$$\mathrm{Y}_{D1} = \left(2h(B + 2A)\gamma^2 - 48(A + B)\gamma X_{12} + \frac{18}{h}(2AX_{11}^2 + 3(A + B)X_{12}^2)\right)\frac{\partial}{\partial\gamma} + \left(\frac{hA\gamma}{3} - X_{12}\right)X_{11}\frac{\partial}{\partial X_{11}} + \left(AX_{11}^2 + BX_{12}^2 - \frac{hB\gamma}{3}X_{12}\right)\frac{\partial}{\partial X_{12}},$$

and a constant of motion

$$J_{D1} = \left[(2A^2 - B(A - 2))\left(2(A - 1)X_{11}^2 + (2A - B - 2)X_{12}^2\right) + 2AC(2A - B - 2)X_{11}^{-2B/A}\right]X_{11}^{-2A+B}.$$

For constitutive functions of the form (72) with $B/A > 0$, for initial conditions $X_{11}(0) = X_{12}(0) = 0$, the ODE system (38) evidently yields

$$X_{11} \equiv 0, \quad X_{12} = \frac{h}{3}\gamma,$$

which does not correspond to a solution of physical interest. However, when $B/A < 0$, there exist physically acceptable solutions; an example is given in Fig. 7.

Case (D2):

$$f(X_{11}, X_{12}) = \frac{h}{3}\,\frac{1}{X_{11} + X_{12}^3 g(X_{11})}, \tag{73}$$

where $g(X_{11})$ is an arbitrary function.

In this case, the system (38) admits a projective symmetry

$$Y_{D2} = \frac{3}{2h}\left(X_{11}^2 - 2X_{12}^2\right)\frac{\partial}{\partial\gamma} - X_{12}^2\frac{\partial}{\partial X_{12}},$$

and a constant of motion

$$J_{D2} = \frac{1}{X_{12}} + \int g(X_{11})dX_{11}.$$

Under the initial conditions $X_{11}(0) = X_{12}(0) = 0$, the conserved quantity J_{D2} requires singular behaviour of $g(X_{11})$ at $X_{11} = 0$.

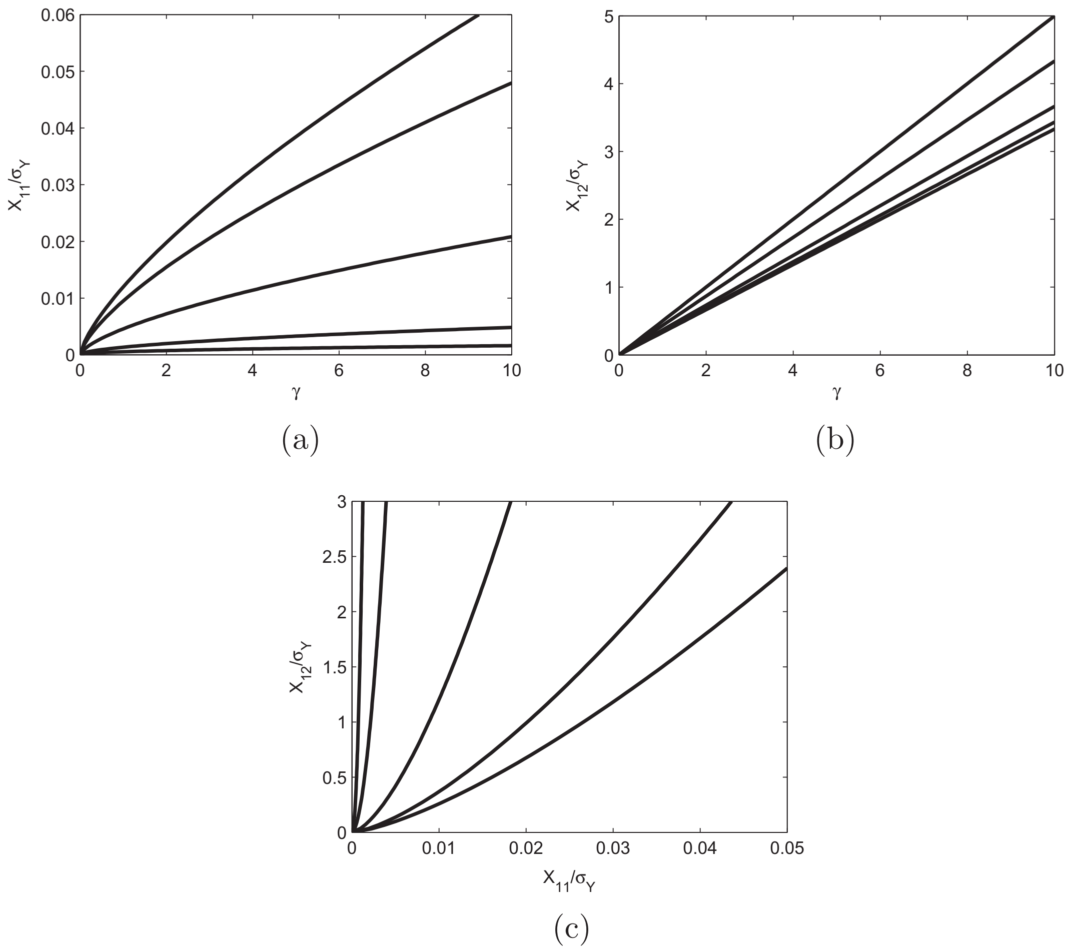

Fig. 7. Sample plots of back stress components X_{11}/σ_Y (a) and X_{12}/σ_Y (b) as functions of γ, and a phase diagram (c), in the case (72). The parameter values used are $A = 3.9\sigma_Y$, $B = -0.4875\sigma_Y$, $C = 0$, $h/\sigma_Y = 1$, 1.031.11.31.5 (from top to bottom in (a), from bottom to top in (b), and from right to left in (c)).

3.5. Discussion

The classification of the constitutive function $c_1(X_{11}, X_{12})$ according to the symmetries and integrating factors inherent to the BVP of simple shear has provided several cases of interest that do exhibit responses a priori satisfying the expected material's response. As previous results have shown, there are nevertheless still many possibilities to formulate the objective rate of the back stress in Dafalias model according to the expression of the coefficient c_1 versus the strain invariants, so one can try to further reduce those cases according to an adequate argumentation extending the few criteria listed in the beginning of Section 3.3.

The fundamental principles at the root of plasticity theories will be recalled, in order to provide more criteria for the selection of the objective rates. Two main issues of concern are stability of the material behavior and non-equilibrium thermodynamics.

Amongst the fundamental principles of thermodynamics, the second principle states a condition to be satisfied for the evolution of irreversible processes, in terms of a state function called entropy, which shall tend to a maximum (along a non-equilibrium path of the system) for isolated systems at equilibrium. A wider statement for non-isolated systems relies on the entropy production, which is the true measure of irreversibility and shall accordingly be nonnegative.

The internal entropy production is the product of the affinity with the associated flux, as exposed in Prigogine and Chanu (1968); in the present model, it reduces to $\sigma_S = (\boldsymbol{\sigma} - \mathbf{X}) : \mathbf{D}^p$, due to the fact that the flux variable $\dot{\boldsymbol{\alpha}}$ dual to the back stress is given from the normality condition $\dot{\boldsymbol{\alpha}} = \dot{\lambda}\partial q/\partial \mathbf{X}$. Since the yield function only depends on $\boldsymbol{\sigma}, \mathbf{X}$ through their difference, we obtain

$$\dot{\boldsymbol{\alpha}} = \dot{\lambda}\frac{\partial q(\boldsymbol{\sigma} - \mathbf{X})}{\partial \mathbf{X}} = -\dot{\lambda}\frac{\partial q(\boldsymbol{\sigma} - \mathbf{X})}{\partial \boldsymbol{\sigma}} = \mathbf{D}^p.$$

When applied to the present situation of simple shear with imposed constant shear strain rate, the internal entropy production writes

$$\sigma_S = (\sigma_{12} - X_{12})D^p_{12} = |\dot{\gamma}|\frac{\sigma_Y}{\sqrt{3}} \geqslant 0.$$

This condition is evidently satisfied.

Drucker proposed to classify materials by his well-known postulate (Drucker, 1951; Wu, 2005). Materials obeying Drucker's postulate can then be studied by a theory of plasticity that can be built up based on that postulate. Three classes of materials may be considered: the strain-hardening material, the perfectly plastic material, and the strain-softening material. Generally, strain-hardening materials satisfy the Drucker's stability postulate (Drucker, 1951; Wu, 2005), which consists of two parts: the *stability in the small* and the *stability in the large*, as described by the following statements:

- *The stability in the small:* The work done by an *external agency*, which slowly applies an additional set of forces to the already stressed material, over the displacement it produces, is positive. This condition is expressed as $\dot{\boldsymbol{\sigma}} : \mathbf{D}^p \geqslant 0$.
- *The stability in the large:* The net work performed by the *external agency* over the *cycle* of application and removal is positive, if plastic deformation has occurred in the cycle. This condition is expressed by the formula $(\boldsymbol{\sigma} - \boldsymbol{\sigma}^*) : \mathbf{D}^p \geqslant 0$, where $\boldsymbol{\sigma}^*$ is a stress state within or on the yield surface at a given time, and $\boldsymbol{\sigma}$ the stress state on the yield surface at subsequent time (this is usually recognized as Hill's principle).

Drucker's postulate implies as a corollary that the yield surface is convex and that the plastic strain increment is normal to the yield surface. Without the use of this postulate, these two effects would have been two separate assumptions. Drucker's stability postulate in the small, for simple shear, has the form

$$\dot{\boldsymbol{\sigma}} : \mathbf{D}^p \geqslant 0,$$

which implies that $\dot{X}_{12}$ and $\dot{\gamma}$ have the same sign. In the present case of positive $\dot{\gamma}, \dot{X}_{12}$ shall be positive for all times, i.e., the shear component X_{12} should be an increasing function of time for the material behavior to be stable. However, one shall allow for unstable behaviors, as well as for transitions from stable to unstable responses which may occur when some material parameters or parameters related to the constitutive function $f(X_{11}, X_{12})$ change. Accordingly, one may be able to tune the response from stable to unstable by changing those parameters.

In almost all the cases considered above, the shear component X_{12} was an increasing function of time. In cases 5 and 6, a transition from a stable to an unstable regime occurred when the ratio h/σ_Y was decreased. In case 5, the unstable behavior corresponds to a part of a center in the phase portrait of the dynamical system (Fig. 3(c)); complete centers whereby the system evolves along a closed loop in the (X_{11}, X_{12})- plane would be observed if a positive initial value of X_{12} is chosen. Stable regimes correspond to open phase trajectories.

In the following Table 1, we classify the obtained responses according to the above-discussed criteria. The following remarks are related to this table. The subcase 2a leads to an oscillatory behavior of $X_{12}(\gamma)$, which is non-physical. In case 2b, $X_{11}(\gamma)$ fails to exist for all shear values. Cases 1, 5, 6 satisfy all listed criteria. For case 5, the Drucker's stability criterion gives a further constraint on the arbitrary function $Q_5(X_{11})$, according to whether one has a stable or unstable behavior. In particular, from the second equation of (43), one has

Table 1
Classification of the form of constitutive function $f(X_{11}, X_{12})$ according to the fundamental principles of TIP. Specific case $D^p_{12} = \dot{\gamma}/2 \geqslant 0$.

Case	Form of $f(X_{11}, X_{12})$	Strength of plastic spin c_1	Possibility of transition from stable ($\dot{X}_{12} \geqslant 0$) to unstable states ($\dot{X}_{12} \leqslant 0$)	Availability of solutions with physical behavior
1	$\frac{K}{X_{11}}$	$\frac{X_{11} - K}{2X_{11}^2}$	No	Yes
2a	$\frac{\sqrt{N^2 + M^2X_{11}^2}}{X_{11}}$	$\frac{X_{11} \mp \sqrt{N^2 + M^2X_{11}^2}}{2X_{11}^2}$	Yes	No: oscillations in X_{12}
2b	$\frac{\sqrt{N^2 - M^2X_{11}^2}}{X_{11}}$	$\frac{X_{11} \mp \sqrt{N^2 - M^2X_{11}^2}}{2X_{11}^2}$	Yes, when changing parameter $\frac{1}{M}\left(\frac{h}{3N} - 1\right)$	No: $X_{11}(\gamma)$ fails to exist for large shear values
3a	$\frac{1}{X_{11}(AX_{12} + B)}$	$\frac{X_{11}(AX_{12} + B) - 1}{2X_{11}^2(AX_{12} + B)}$	No	Yes
4	$\frac{Q_4(X_{11})}{X_{12}}$	$\frac{X_{12} - Q_4(X_{11})}{2X_{11}X_{12}}$	No	Yes
5	$Q_5(X_{11})$	$\frac{1 - Q_5(X_{11})}{2X_{11}}$	Yes, when changing material parameter h/σ_Y and constitutive parameters	Yes
6	$\frac{Q_6(X_{12})}{X_{11} + A}$	$\frac{X_{11} + A - Q_6(X_{12})}{2X_{11}(X_{11} + A)}$	Yes	Yes
(D2)	$\frac{1}{3}\frac{h}{X_{11} + X_{12}^3 g(X_{11})}$	$\frac{3(X_{11} + X_{12}^3 g(X_{11})) - h}{6X_{11}(X_{11} + X_{12}^3 g(X_{11}))}$	Yes	Yes

$$\frac{dX_{12}}{d\gamma} = \frac{h}{3} - X_{11}Q_5(X_{11}),$$

which requires that the right-hand side $X_{11}Q_5(X_{11}) < h/3$ for stable configurations with $X_{11}, X_{12}, dX_{12}/d\gamma \geqslant 0$. The reverse inequality holds in the unstable regime.

Dafalias (1985a) used a model for the constitutive spin with a constant c_1 parameter,

$$\boldsymbol{\omega} = \mathbf{W} - \mathbf{W}^p = \mathbf{W} - \eta[\mathbf{X}, \mathbf{D}],$$

where η is a parameter describing the strength of plastic spin. This case appears as a specific subcase of Case 5, corresponding to $f(X_{11}) = Q_5(X_{11}) = 1 - 2\eta X_{11}$, resulting in a constant strength of plastic spin.

Since the constitutive model for kinematic hardening considered in the current work is a rate-independent model, and thus does not explicitly involve time, the simple shear process being analyzed is an equilibrium path. It follows that the principle of maximum entropy production (Ziegler's principle, see Ziegler, 1963; Ziegler, 1983; Martyushev and Seleznev, 2006) does not apply. An extension of the current framework to the consideration of rate-dependent constitutive laws would make the Ziegler's principle applicable. In particular, this principle would pose constraints on the form of any rate dependent constitutive model for kinematic hardening, and could help shed more light onto the fundamental principles of irreversible thermodynamics.

4. Conclusions

In the present work, we have extended the model of the constitutive spin introduced by Dafalias (1985b) to describe the evolution of the orientation of the material texture, considering a rigid plastic constitutive model with linear kinematic hardening. The constitutive spin is an essential ingredient necessary to express the objective rate of the back stress. A more general form of the constitutive spin using representation theorems for second order tensors has been proposed in Eqs. (9) and (10), involving a priori arbitrary functions of the tensor invariants. Since substantial freedom of choice exists for these constitutive functions, a rational methodology based on the structure of the underlying equations (symmetries and integrating factors) is proposed to make appropriate choices. Independently from their aesthetic attraction, symmetries and conserved quantities intrinsically contained in physical systems allow for a concise description their invariance properties (geometrical, temporal, or under a given transformation of a set of relevant variables), and provide ways of systematic prediction of the various phenomena that may occur in these systems.

Focusing on planar simple shear, the constitutive functions were classified according to continuous symmetry groups and integrating factors admitted by the system of differential Eqs. (24) for the back stress components. Each form of the constitutive function corresponds to an the objective rate of the back stress given by (11). Exact and/or numerical solutions for stress versus time were obtained for some objective rates of the back stress. The classification of the constitutive function of the model according to the symmetries and first integrals inherent to the BVP of simple shear resulted in isolating six main cases of interest, for which the temporal evolution of the normal and shear stresses exhibits a physically sound behavior, in particular, no oscillatory behavior for the shear component (these cases are listed in Table 1). The presented approach relying on the internal ODE structure provides a powerful and rational methodology for the classification of the material behavior. The partial symmetry/integrating factor classification performed in the current work might not reveal all cases of interest, however it was indeed successful in revealing several cases with additional mathematical structure. In the present case, the calculation of dynamical invariants in terms of the components of the back stress tensors and of the shear strain allowed to directly evaluate the dynamical response of the material, in terms of the phase portrait drawn in the space of the two independent components of the back stress tensor. This is one of the principal novel aspects of this work. Despite its apparent simplicity, the simple shear test proves to be rich enough in terms of the diversity of material responses obtained from the symmetry analysis.

The application of the fundamental principles of irreversible thermodynamics, namely the positivity of the internal entropy production and material stability in the sense of Drucker, provides a filtering mechanism for the constitutive models revealed by additional symmetries and integrating factors. Constitutive models given by cases 5, 6 and (D2) in Table 1 appear to be most physically relevant. In particular, the model corresponding to case 5 is a generalization of Dafalias spin model (5), in the sense that the material parameter within the expression for the plastic spin now depends on the back stress. The constitutive models suggested in the current paper need to be validated experimentally in future work.

The analysis presented in the current paper can be extended to include more general kinematic hardening rules, such as, for example, the Armstrong–Frederick rule (Appendix A), for which the system of Eqs. (23), (24) is appended by terms linear in the back stress components on the right hand side, see (A.5). The analysis of this more complex system is under way. As briefly outlined in the Discussion section, the principle of maximum entropy production could bring further constraints on the form of the constitutive functions when more general and physically more realistic constitutive models for kinematic hardening, including rate-dependent ones, are considered.

We note that the case of simple shear considered in the current work leads to a nil coefficient c_2 in (9), which will not be the case for more complex loadings, including cyclic loading paths. Such extensions will be addressed in the future work. In more general settings, the argumentation based on irreversible thermodynamics will lead to different criteria to be satisfied by the back stress tensor.

The predictive methodology based on the internal mathematical structure of the model shall be extended in future to treat more complex situations of inelastic material behaviors. As the present work suggests, it is further believed that symmetry methods can be helpful to discriminate amongst the existing objective rates, and possibly lead to the construction of new objective rates in elastoplasticity. Those considerations open new promising avenues in the analysis of plasticity models based on a combination of the thermodynamics of irreversible processes and the symmetry framework.

Appendix A. Dynamic ODEs for the back stress tensor components in the case of the Armstrong–Frederick hardening rule

Instead of the linear Prager's type kinematic hardening rule (12), one may adopt a more general Armstrong–Frederick hardening rule (Frederick and Armstrong, 2007) given by

$$\hat{\mathbf{X}} = \frac{2h}{3}\mathbf{D}^p - \tilde{h}\mathbf{X}D_e^p. \tag{A.1}$$

In (A.1), the equivalent plastic strain rate D_e^p is given by (13), and $h, \tilde{h} = \text{const}$. For the case of the simple shear considered in Section 2.2, from (16), one has

$$D_e^p = \frac{2}{\sqrt{3}}\omega,$$

where $\omega = \omega(t)$ is related to the shear rate by $d\gamma/dt = 2\omega(t)$. The analog of the tensor ODE (20) for the Armstrong-Frederick hardening rule therefore writes as

$$\dot{\mathbf{X}} = \boldsymbol{\omega}\mathbf{X} - \mathbf{X}\boldsymbol{\omega} + \frac{2h}{3}\mathbf{D}^p - \frac{2}{\sqrt{3}}\tilde{h}\omega\mathbf{X}. \tag{A.2}$$

The corresponding equations for the components of the back stress tensor are given by

$$\begin{aligned} &\frac{dX_{11}}{d\gamma} - fX_{12} = \frac{1}{\sqrt{3}}\tilde{h}X_{11}, \\ &\frac{dX_{22}}{d\gamma} + fX_{12} = \frac{1}{\sqrt{3}}\tilde{h}X_{22}, \\ &\frac{dX_{12}}{d\gamma} + \frac{1}{2}f(X_{11} - X_{22}) = \frac{h}{3} + \frac{1}{\sqrt{3}}\tilde{h}X_{12}. \end{aligned} \tag{A.3}$$

Adding the first two equations of (A.3), one obtains

$$\frac{d}{d\gamma}(X_{11} + X_{22}) = \frac{1}{\sqrt{3}}\tilde{h}(X_{11} + X_{12}), \tag{A.4}$$

which together with the initial condition $X_{11}(0) = X_{22}(0) = 0$ yields a unique solution $X_{11} + X_{12} \equiv 0$. Finally, the equations for the back stress tensor components for the case of Armstrong-Frederick hardening rule become

$$\frac{dX_{11}}{d\gamma} - fX_{12} = \frac{1}{\sqrt{3}}\tilde{h}X_{11},$$
$$\frac{dX_{12}}{d\gamma} + fX_{11} = \frac{h}{3} + \frac{1}{\sqrt{3}}\tilde{h}X_{12}, \quad (A.5)$$
$$X_{22} = -X_{11}.$$

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