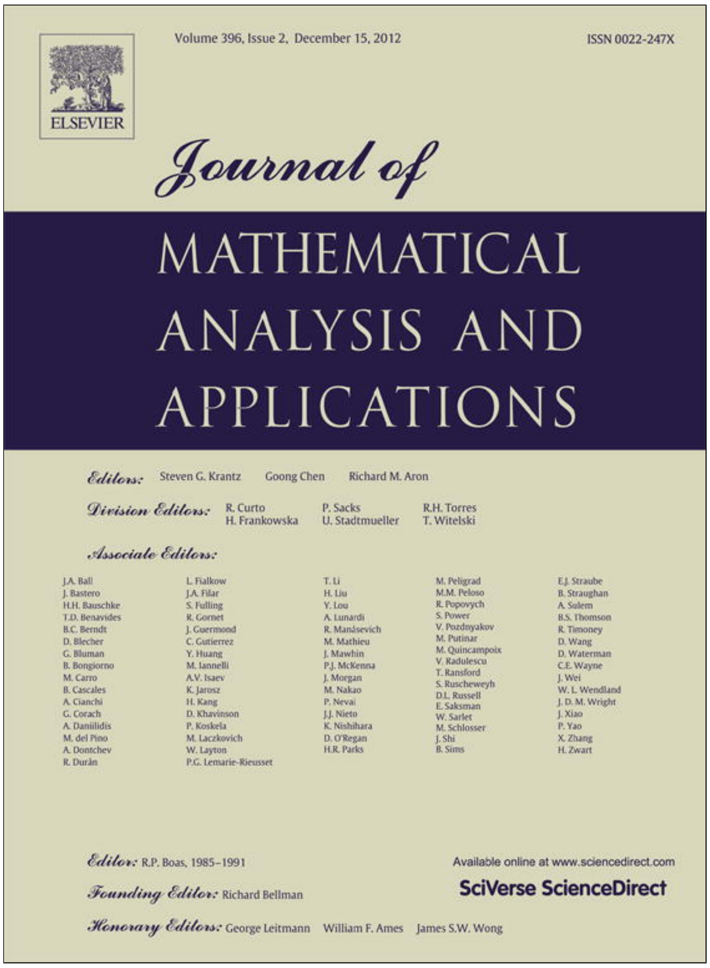
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Symmetry properties of two-dimensional Ciarlet–Mooney–Rivlin constitutive models in nonlinear elastodynamics

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ABSTRACT

Nonlinear dynamic equations for isotropic homogeneous hyperelastic materials are considered in the Lagrangian formulation. An explicit criterion of existence of a natural state for a given constitutive law is presented, and is used to derive natural state conditions for some common constitutive relations.

For two-dimensional planar motions of Ciarlet–Mooney–Rivlin solids, equivalence transformations are computed that lead to a reduction of the number parameters in the constitutive law. Point symmetries are classified in a general dynamical setting and in traveling wave coordinates. A special value of traveling wave speed is found for which the nonlinear Ciarlet–Mooney–Rivlin equations admit an additional infinite set of point symmetries. A family of essentially two-dimensional traveling wave solutions is derived for that case.

1. Introduction

Symmetry requirements form a cornerstone in continuum mechanics. One may classify such symmetries as material symmetries (acting in the reference configuration, they determine what is called the material symmetry group of a given material), and spatial symmetries, which have to be universally satisfied by any theory of finite deformation, and are usually referred to as material frame indifference. Following an early work by Adekele [1], Ericksen [2] was one of the few researchers to develop the idea of general point transformations of both the Lagrangian coordinates and of the fields, thereby extending the concept of material symmetries described by a mere change of the reference configuration. Those generalized coordinate transformations are invariance properties in the sense that they are compatible with the Galilean invariance of the strain energy density function, namely, the material frame indifference is preserved.

Lie symmetry group analysis (e.g., [3]) is an important systematic method for studying systems of differential equations (DEs) and boundary value problems. Applications of Lie symmetry groups include construction of exact (group-invariant, or symmetry-generated) solutions to nonlinear models, and construction of mappings relating differential problems. Knowledge of symmetries of a system of differential equations provides essential information about the internal structure of the system. In particular, the Lie algebra of generators of point symmetries admitted by a given system of equations is coordinate-independent, i.e., invariant under point transformations acting on the variables of the system (including transformations to curvilinear coordinates, as well as more general transformations). From a more formal point of view, Lie symmetries constitute a guide in the Lagrangian and Hamiltonian formalisms in continuum mechanics, especially for complex materials endowed with a microstructure [4]. For models that admit a variational formulation, there is a one-to-one correspondence between variational Lie symmetries and local conservation laws through Noether's theorem; for non-variational models, this relation generally does not hold [5].

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For elastostatics, a large number of contributions is available where families of exact solutions corresponding to various physical set-ups are constructed; specialized methods have been developed to treat specific kinds of problems. Examples of such solutions include works by Hill, Ogden, Spencer and their collaborators (see, e.g., [6–9] and references therein).

As mentioned in the recent contribution [10], finding closed-form time-dependent solutions in compressible hyperelasticity is quite challenging. Particular classes of exact solutions have been obtained for several models using continuous symmetry groups. In particular, the classification of Lie point symmetries for 1D and 2D nonlocal elastodynamics appears in [11,12]. Invariant solutions for the radial motions of compressible hyperelastic spheres and cylinders have been obtained in [13,14]. Similarity solutions for the motion of hyperelastic solids have been determined in [15–18]. In [19], Lie symmetry analysis has been performed and invariant solutions have been derived for incompressible neo-Hookean dynamic elasticity equations. One-dimensional wave-type exact solutions for neo-Hookean dynamics have been derived in [20]. General forms of symmetry groups and invariant solutions for dynamic equations for general hyperelastic solids are discussed in [15] and references therein.

In the presence of arbitrary (constitutive) functions and/or parameters in a given system, symmetry and conservation law classifications are performed to isolate special forms of constitutive functions/parameters that yield additional symmetry/conservation law structure. Such special cases often correspond to models that bear particular physical significance, including integrable models and models linearizable by local or nonlocal transformations (see, e.g., [3,10,21]).

The current paper is concerned with dynamic equations for nonlinear isotropic homogeneous hyperelastic materials, in three-dimensional and two-dimensional (planar) settings. After presenting the equations in the general Lagrangian and Eulerian formulation in Section 2, in the subsequent Section 3 we consider various forms of the constitutive function (volumetric strain energy density) used in the literature, in view of the important question of existence of a *natural* (stress-free) state for boundary value problems. Natural state is a physically relevant requirement important for modeling and simulations; it restricts the forms of admissible constitutive functions. A classical result provides that if the reference configuration is a stress-free (natural) state of an isotropic hyperelastic body, then equations of motion cannot be linear [22,23]. However, if residual stresses are allowed, then equations of motion indeed can be linear. Section 3 contains an explicit necessary and sufficient condition on the form of the strain energy density, expressed in terms of invariants, for a natural state to exist. Some classical models are considered against this criterion.

Section 4 is devoted to a brief review of point symmetries and equivalence transformations for partial differential equations (PDEs).

In Section 5, we derive a set of equivalence transformations of the two-dimensional Ciarlet–Mooney–Rivlin dynamic elasticity equations. The obtained equivalence transformations are subsequently used to reduce the number of parameters in the constitutive model from four to three.

In Section 6, we classify point symmetries of the two-dimensional Ciarlet–Mooney–Rivlin dynamic equations. The classification problem leads to a large system of determining equations containing nearly a thousand linear partial differential equations for the unknown symmetry components. Case splitting is performed and point symmetries are computed using the `Maple rifsimp` routine and the symbolic software package `GeM` [24]. The obtained symmetry structure provides valuable information for future research towards the computation of exact solutions and conservation laws admitted by the Ciarlet–Mooney–Rivlin model. In particular, new cases that yield infinite sets of point symmetries are found.

The traveling wave ansatz of the type $f(x,t) = h(x - st)$, where the observer is moving with a constant speed s, is an important reduction of nonlinear models that admit space and time translations. Traveling wave solutions play a critical role in the description of many nonlinear phenomena, such as solitons (e.g., [25]). Traveling wave ansatz in a general three-dimensional hyperelastic setting was studied in [26], where a number of exact solutions has been derived. Interesting results pertaining to traveling waves in elastic Mooney–Rivlin rods were obtained in [27] and works it refers to.

In Section 7, we apply the traveling wave ansatz to the two-dimensional compressible Ciarlet–Mooney–Rivlin model; this leads to a PDE system of a rather special mathematical structure. The classification of point symmetries of the resulting system yields a number of nontrivial cases that admit additional symmetries. In particular, in the case when a special relation holds between the wave speed and the parameters of the constitutive model, the equations admit an additional sets of symmetries, including an infinite family of symmetries for the case of constant body density. This special value of the wave speed in the nonlinear model is shown to be equal to the traveling wave speed of a certain pair of linear wave equations. Finally, a sample family of exact two-dimensional traveling wave-type solutions is presented.

2. Equations of motion, constitutive relations, and boundary value problems for hyperelastic materials

In order to set the stage, in the current section, we recall the main ingredients of hyperelastic models.

Within the paper, boldface notation will be used to denote vector and tensor quantities. Partial derivatives are often denoted by subscripts: $\partial u/\partial x \equiv u_x$.

2.1. The material picture

Consider a solid body that at time $t = 0$ occupies the spatial region $\overline{\Omega}_0 \subset \mathbb{R}^3$ (reference, or Lagrange configuration). Here Ω_0 is an open bounded connected set having a Lipschitz boundary [28].

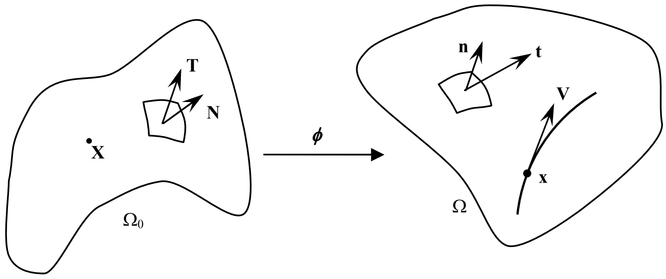

Fig. 1. Material and Eulerian coordinates.

The actual position $\mathbf{x}$ of a material point labeled by $\mathbf{X} \in \overline{\Omega}_0$ at time t is given by

$$\mathbf{x} = \boldsymbol{\phi}\left(\mathbf{X}, t\right), \qquad x^i = \phi^i\left(\mathbf{X}, t\right).$$

Coordinates $\mathbf{X}$ in the reference configuration are commonly referred to as Lagrangian coordinates, and actual coordinates $\mathbf{x}$ as Eulerian coordinates. The deformed body occupies an Eulerian domain $\overline{\Omega} = \boldsymbol{\phi}(\overline{\Omega}_0) \subset \mathbb{R}^3$ (Fig. 1). The velocity of a material point $\mathbf{X}$ is given by

$$\mathbf{v}\left(\mathbf{X}, t\right) = \frac{d\mathbf{x}}{dt} \equiv \frac{d\boldsymbol{\phi}}{dt}.$$

The mapping $\boldsymbol{\phi}$ must be sufficiently smooth (the regularity conditions depending on the particular problem). The Jacobian matrix of the coordinate transformation is given by the deformation gradient

$$\mathbf{F}(\mathbf{X}, t) = \nabla\boldsymbol{\phi}, \tag{1}$$

which is an invertible matrix with components

$$F^i_{\ j} = \frac{\partial \phi^i}{\partial X^j} = F_{ij}. \tag{2}$$

(Throughout the paper, we use Cartesian coordinates and flat space metric tensor $g^{ij} = \delta^{ij}$, therefore indices of all tensors can be raised or lowered freely as needed.) The transformation satisfies the orientation preserving condition

$$J = \det \mathbf{F} > 0.$$

Forces and stress tensors

By the well-known Cauchy theorem, the force (per unit area) acting on a surface element S within or on the boundary of the solid body is given in the Eulerian configuration by

$$\mathbf{t} = \boldsymbol{\sigma}\mathbf{n},$$

where $\mathbf{n}$ is a unit normal, and $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{x}, t)$ is Cauchy stress tensor (see Fig. 1). The Cauchy stress tensor is symmetric: $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$, which is a consequence of the conservation of angular momentum. For an elastic medium undergoing a smooth deformation under the action of prescribed surface and volumetric forces, the existence and uniqueness of the Cauchy stress $\boldsymbol{\sigma}$ follows from the conservation of momentum (cf. [29, Section 2.2]). The force acting on a surface element S_0 in the reference configuration is given by the stress vector

$$\mathbf{T} = \mathbf{P}\mathbf{N},$$

where $\mathbf{P}$ is *the first Piola–Kirchhoff tensor*, related to the Cauchy stress tensor through

$$\mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^{-T}. \tag{3}$$

In (3), $(F^{-T})_{ij} \equiv (F^{-1})_{ji}$ is the transpose of the inverse of the deformation gradient.

Hyperelastic materials

A *hyperelastic* (or Green elastic) material is an ideally elastic material for which the stress–strain relationship follows from a strain energy density function; it is the material model most suited to the analysis of elastomers. In general, the response of an elastic material is given in terms of the first Piola–Kirchhoff stress tensor by $\mathbf{P} = \mathbf{P}\left(\mathbf{X}, \mathbf{F}\right)$. A *hyperelastic* material assumes the existence of a scalar valued volumetric strain energy function $W = W\left(\mathbf{X}, \mathbf{F}\right)$ in the reference configuration, encapsulating all information regarding the material behavior, and related to the stress tensor through

$$\mathbf{P} = \rho_0 \frac{\partial W}{\partial \mathbf{F}}, \qquad P^{ij} = \rho_0 \frac{\partial W}{\partial F_{ij}}, \tag{4}$$

where $\rho_0 = \rho_0(\mathbf{X})$ is the time-independent body density in the reference configuration. The actual density in Eulerian coordinates $\rho = \rho(\mathbf{X}, t)$ is time-dependent and is given by

$$\rho = \rho_0 / J.$$

2.2. Equations of motion

Equations of motion of hyperelastic material can be formulated in Lagrangian or Eulerian framework. In the Lagrangian (material) framework, the unknowns are the positions $\mathbf{x} = \boldsymbol{\phi}(\mathbf{X}, t)$ of material points labeled by $\mathbf{X}$. The full system of equations of motion is then given by

$$\rho_0 \mathbf{x}_{tt} = \mathrm{div}_{(X)} \mathbf{P} + \rho_0 \mathbf{R}, \tag{5a}$$

$$\mathbf{F}\mathbf{P}^T = \mathbf{P}\mathbf{F}^T, \tag{5b}$$

$$\mathbf{P} = \rho_0 \frac{\partial W}{\partial \mathbf{F}}. \tag{5c}$$

Eq. (5a) expresses the conservation of momentum for an infinitesimal volume in the reference configuration; $\mathbf{R} = \mathbf{R}(\mathbf{X}, t)$ is the total body force per unit mass; divergence is taken with respect to material coordinates and is given by

$$(\mathrm{div}_{(X)} \mathbf{P})^i = \frac{\partial P^{ij}}{\partial X^j}.$$

Eq. (5b) is equivalent to the Cauchy stress tensor symmetry condition $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$ and expresses the conservation of angular momentum. Eq. (5c) is the hyperelastic constitutive law (4). Particular forms of the strain energy density function W used in models are discussed in Section 2.3.

Remark 1. It is important to note that for the case of zero forcing or when external forces are potential, equations of motion (5a) follow from a variational formulation (see the Appendix).

Remark 2. For *incompressible* materials, the formula (5c) is replaced by

$$\mathbf{P} = -p\ \mathbf{F}^{-T} + \rho_0 \frac{\partial W}{\partial \mathbf{F}}, \tag{6}$$

or, in components,

$$P^{ij} = -p\ (F^{-1})^{ji} + \rho_0 \frac{\partial W}{\partial F_{ij}}, \tag{7}$$

where $p = p\,(\mathbf{X}, t)$ is the hydrostatic pressure [29].

In the Eulerian configuration, the natural unknowns are the components of velocity $\mathbf{v}^e(\mathbf{x}, t) = \mathbf{v}(\mathbf{X}, t)$ of points in the moving solid body. The conservation of momentum (5a) in Eulerian coordinates takes the form

$$\rho \mathbf{v}^e_t = \mathrm{div}_{(x)} \boldsymbol{\sigma} + \rho \mathbf{r}, \tag{8}$$

where $\mathbf{r} = \mathbf{r}(\mathbf{x}, t)$ are body forces (per unit mass). Eq. (5b) in Eulerian framework explicitly takes the form of the Cauchy stress symmetry condition $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$. The constitutive law (5c) can be appropriately rewritten in Eulerian variables.

For one-dimensional elasticity problems, all vector and tensor quantities become scalars, and the hyperelastic constitutive relation (5c) is rewritten in Eulerian variables in a straightforward way. In that case, the Eulerian and the Lagrangian system can be shown to be nonlocally related; they can be used independently to obtain complementary results, for example, in the computation of admitted symmetries and conservation laws [10].

2.3. Constitutive relations

In the current paper, the attention is restricted to a rather wide class of models known as *isotropic homogeneous hyperelastic materials*. An elastic material is *homogeneous* if $\mathbf{P}$ does not explicitly depend on the coordinates $\mathbf{X}$. An elastic material is *isotropic* if it is invariant under rotations: $\mathbf{P}(\mathbf{X}, \mathbf{F}) = \mathbf{P}(\mathbf{X}, \mathbf{QF})$ for any rotation matrix $\mathbf{Q}$.

Consider an isotropic homogeneous hyperelastic material with strain energy density function $W = W\,(\mathbf{X}, \mathbf{F})$. Since $J = \det \mathbf{F} > 0$, one can use polar decomposition, $\mathbf{F} = \mathbf{RU} = \mathbf{VR}$, where $\mathbf{R}$ is a rotation matrix, and $\mathbf{U}, \mathbf{V}$ are the right and the left stretch tensors, given by symmetric positive-definite matrices. Matrices $\mathbf{U}, \mathbf{V}$ are similar and thus share the set of eigenvalues $\lambda_1, \lambda_2, \lambda_3$, called *principal stretches*. It has been shown [29] that for such materials, the strain energy function is given by

$$W = \Phi(\lambda_1, \lambda_2, \lambda_3). \tag{9}$$

Consider the left Cauchy–Green strain tensor $\mathbf{B}$ and the right Cauchy–Green strain tensor $\mathbf{C}$ defined by

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T, \qquad B^{ij} = F^i{}_k F^j{}_k, \qquad \mathbf{C} = \mathbf{F}^T\mathbf{F}, \qquad C_{ij} = F^k{}_i F^k{}_j. \tag{10}$$

Tensors $\mathbf{B}$ and $\mathbf{C}$ have the same sets of eigenvalues, given by the squares of principal stretches: $\lambda_1^2, \lambda_2^2, \lambda_3^2$. The three principal invariants of the Cauchy–Green tensors $\mathbf{B}$ and $\mathbf{C}$ are given by

$$I_1 = \mathrm{Tr}\mathbf{B} = F^i{}_k F^i{}_k, \qquad I_2 = \frac{1}{2}[(\mathrm{Tr}\mathbf{B})^2 - \mathrm{Tr}(\mathbf{B}^2)] = \frac{1}{2}(I_1^2 - B^{ik}B^{ki}), \qquad I_3 = \det \mathbf{B} = J^2. \tag{11}$$

Table 1
Neo-Hookean and Mooney–Rivlin constitutive models.

Type	Neo-Hookean	Mooney–Rivlin
Standard [29]	$W = aI_1,\ a > 0.$	$W = aI_1 + bI_2,\ a, b > 0$
Generalized [30]	$W = a\bar{I}_1 + c(J-1)^2,\ a, c > 0.$	$W = a\bar{I}_1 + b\bar{I}_2 + c(J-1)^2,\ a, b, c > 0$
Generalized (Ciarlet) "compressible" [28]	$W = aI_1 + \Gamma(J),\ \Gamma(q) = cq^2 - d\log q,\ a, c, d > 0.$	$W = aI_1 + bI_2 + \Gamma(J)\Gamma(q) = cq^2 - d\log q,\ a, b, c, d > 0$

Moreover,

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \qquad I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_1^2\lambda_3^2, \qquad I_3 = \lambda_1^2\lambda_2^2\lambda_3^2. \tag{12}$$

In terms of the invariants I_1, I_2, I_3, the strain energy density of an isotropic homogeneous hyperelastic material is given by

$$W = \Phi(\lambda_1, \lambda_2, \lambda_3) = U(I_1, I_2, I_3) = \bar{U}(\bar{I}_1, \bar{I}_2, \bar{I}_3), \tag{13}$$

where

$$\bar{I}_1 = J^{-2/3}I_1, \qquad \bar{I}_2 = J^{-4/3}I_2, \qquad \bar{I}_3 = J$$

is an alternative set of invariants that is sometimes used [30].

In the current paper, we restrict our attention to Hadamard materials [29,31], for which the constitutive relation is commonly written as

$$W = a(I_1 - 3) + b(I_2 - 3) + H(I_3), \quad a, b > 0, \tag{14}$$

where the constant "−3" can be omitted without loss of generality.

Some commonly used forms of the Hadamard-type constitutive relations, including the neo-Hookean and Mooney–Rivlin models and their generalizations, are listed in Table 1.

In particular, the usual two-parameter Mooney–Rivlin model [29,31] is given by

$$W = aI_1 + bI_2, \quad a, b > 0. \tag{15}$$

This constitutive relation is often considered valid for rubber-like materials undergoing strains less than 100%; the coefficients a, b are determined experimentally. In the current paper, we study a generalization of (15), which we refer to as *the Ciarlet–Mooney–Rivlin model*.

$$W = aI_1 + bI_2 - cI_3 - \frac{1}{2}d\log I_3, \quad a > 0,\ b, c, d \geq 0. \tag{16}$$

(The relation (16) is called simply the "compressible" Mooney–Rivlin model in [28].)

Numerous other constitutive relations are used in the literature, including, for example, those for Ogden materials and generalized Blatz–Ko foam rubbers (see, e.g., [29,31]).

General form of the stress tensor

It can be shown that the stress of an isotropic elastic material and the Cauchy–Green tensors are related as follows.

Theorem 1. *For an isotropic elastic material, the second Piola–Kirchhoff stress tensor has the form*

$$\mathbf{S} \equiv \mathbf{F}^{-1}\mathbf{P} = 2\rho_0\left[\alpha_0\mathbf{I} + \alpha_1\mathbf{C} + \alpha_2\mathbf{C}^2\right], \tag{17}$$

where $\mathbf{I}$ *is the unit tensor, and* $\alpha_i(I_1, I_2, I_3)$ *are functions of the three principal invariants (11) of the Cauchy–Green tensors.*

The proof of Theorem 1 appears in [29]. The following corollary is an immediate consequence of (17).

Corollary 1. *For an isotropic elastic material, the symmetry condition* $\mathbf{F}\mathbf{P}^T = \mathbf{P}\mathbf{F}^T$ *(5b) is identically satisfied.*

Remark 3. One cannot directly use Eq. (17) as a general constitutive equation for an isotropic hyperelastic material, since functions $\alpha_i = \alpha_i(I_1, I_2, I_3)$, $i = 0, 1, 2$, are not independent. In fact, the expression (17) follows from the constitutive relations (5c); one explicitly finds (cf. (13))

$$\alpha_0 = \frac{\partial U}{\partial I_1} + I_1\frac{\partial U}{\partial I_2} + I_2\frac{\partial U}{\partial I_3}; \qquad \alpha_1 = -\frac{\partial U}{\partial I_2} - I_1\frac{\partial U}{\partial I_3}, \qquad \alpha_2 = \frac{\partial U}{\partial I_3}. \tag{18}$$

Indeed, solving for the partial derivatives of U, one has

$$\frac{\partial U}{\partial I_1} = \alpha_0 + I_1(\alpha_1 + I_1\alpha_2) - I_2\alpha_2, \qquad \frac{\partial U}{\partial I_2} = -\alpha_1 - I_1\alpha_2, \qquad \frac{\partial U}{\partial I_3} = \alpha_2.$$

It follows that additional compatibility conditions

$$\frac{\partial^2 U}{\partial I_i \partial I_j} = \frac{\partial^2 U}{\partial I_j \partial I_i}, \quad i, j = 1, 2, 3, \ i \neq j \tag{19}$$

have to be satisfied. The conditions (19) in turn induce similar compatibility conditions for α_i, $i = 0, 1, 2$. Hence the choice of α_i is restricted to triplets satisfying these compatibility conditions.

As the simplest example, considering the standard neo-Hookean constitutive relation $W = aI_1$, the Piola–Kirchhoff stress tensor takes the form

$$P^{ij} = \rho_0 \frac{\partial W}{\partial F_{ij}} = 2\rho_0 F_{ij}, \tag{20}$$

which also follows directly from (17) and (18) with $W \equiv U(I_1) = I_1, \alpha_0 = 1, \alpha_1 = \alpha_2 = 0$.

3. Constitutive relations and natural state

In many elasticity problems, the reference (Lagrangian) configuration $\overline{\Omega}_0$ is a *natural state*, i.e., zero displacement implies zero stress. In such cases, the constitutive relation $W = W(\mathbf{F})$ to be used should be compatible with the natural state. [Otherwise, a well-posed boundary value problem cannot be formulated.] It follows that when $\mathbf{x} = \mathbf{X}$, i.e., $\mathbf{F} = \mathbf{I} = Id$, $J = 1$, the Cauchy stress should vanish: $\boldsymbol{\sigma} = 0$.

As an example of a constitutive relation that fails to correspond to a natural state, consider a standard neo-Hookean model (Table 1). From (20) it follows that when $\mathbf{F} = \mathbf{I}$, the residual stress

$$\boldsymbol{\sigma} = J^{-1}\mathbf{P}\mathbf{F}^T = 2\rho_0 \mathbf{I}$$

is nonzero, hence the no-displacement state can only be supported by external forces.

In general, it can be proven that if the reference configuration is a stress-free (natural) state of an isotropic hyperelastic body, then Cauchy and first Piola–Kirchhoff stress tensors cannot be linear in $\mathbf{F}$, and thus equations of motion (5) *cannot be linear* [22,23]. However, if residual stresses are allowed, then equations of motion indeed can be linear. A simplest example of linear equations of motion is provided by the same neo-Hookean model.

In general, it is possible to specify the general requirement on the hyperelastic constitutive laws (13) in order for it to satisfy the natural state assumption.

Theorem 2. *A hyperelastic material with the constitutive law given by* (13) *has zero residual stress (i.e., the reference configuration is natural) if and only if*

$$\frac{\partial U}{\partial I_1} + 2\frac{\partial U}{\partial I_2} + \frac{\partial U}{\partial I_3} = 0 \quad \textit{when } \mathbf{F} = \mathbf{I}. \tag{21}$$

Proof. When $\mathbf{F} = \mathbf{I}$, one has

$$\mathbf{B} = \mathbf{C} = \mathbf{I}, \qquad I_1 = I_2 = 3, \qquad I_3 = 1, \qquad \mathbf{P} = \mathbf{S} = \boldsymbol{\sigma}. \tag{22}$$

Requiring that the right-hand side of (17) vanishes and using (18) and (22), one arrives at the condition (21). □

As an example, considering the Mooney–Rivlin constitutive relation (15), one finds that the condition (21) implies $a + 2b = 0$, which is never satisfied for $a > 0$, $b \geq 0$; hence the classical neo-Hookean and Mooney–Rivlin models indeed do not correspond to a natural state.

Theorem 2 can be used to derive conditions for the existence of a natural state for any constitutive relation of the type (13). In particular, for constitutive functions in the form $W = \Phi(\lambda_1, \lambda_2, \lambda_3)$, one can show using (12) that the condition (21) is equivalent to the vanishing of the three principal Biot stresses t_k:

$$t_k = \frac{\partial \Phi}{\partial \lambda_k} = 0 \quad \text{when } \lambda_1 = \lambda_2 = \lambda_3 = 1; \ k = 1, 2, 3. \tag{23}$$

Further examples of natural state conditions that follow from (21) and (23) for various constitutive models are given in Table 2. In particular, the Ciarlet–Mooney–Rivlin model (16) corresponds to a natural state when

$$a + 2b + c - \frac{1}{2}d = 0. \tag{24}$$

4. Point symmetries and equivalence transformations

In the current section, we briefly review the notions of point symmetries (Section 4.1) and equivalence transformations (Section 4.2) for partial differential equations. For full details, see, e.g., [3].

Table 2
Conditions for the existence of a natural state for some constitutive models.

Type	Strain energy density function W	Natural state condition
Hadamard material (incl. neo-Hookean, Mooney–Rivlin)	$a(I_1-3)+b(I_2-3)+H(I_3)$	$a+2b+H'(1)=0$
Generalized Blatz–Ko rubber model	$\frac{\mu}{2}f\left(I_1-1-\frac{1}{\nu}+\frac{1}{q}I_3^{-q}\right)+\frac{\mu}{2}(1-f)\left(\frac{I_2}{I_3}-1-\frac{1}{\nu}+\frac{1}{q}I_3^{q}\right)$	All admissible $\mu, \nu, f; q\equiv\frac{\nu}{1-2\nu}$
Generalized Mooney–Rivlin material	$a(\bar{I}_1-3)+b(\bar{I}_2-3)+c(J-1)^2$	All admissible a, b, c
Ogden material	$\sum_{i=1}^{M}a_i(\lambda_1^{\alpha_i}+\lambda_2^{\alpha_i}+\lambda_3^{\alpha_i})+\sum_{j=1}^{N}b_j((\lambda_1\lambda_2)^{\beta_j}+(\lambda_2\lambda_3)^{\beta_j}+(\lambda_3\lambda_1)^{\beta_j})+H(\lambda_1\lambda_2\lambda_3)$	All admissible $a_i, b_j, \alpha_i, \beta_j; H'(1)=0$

In Section 5, equivalence transformations of the two-dimensional Ciarlet–Mooney–Rivlin system are used to reduce the number of parameters in the constitutive model. Further, in Section 6, point symmetries of the full two-dimensional Ciarlet–Mooney–Rivlin system are classified. In Section 7, point symmetries of the Ciarlet–Mooney–Rivlin system in traveling wave coordinates are classified.

In order to compute point symmetries, one needs to solve a linear overdetermined system of determining equations for symmetry components. Determining equations often involve hundreds to thousands of linear PDEs. In order to solve such systems effectively, symbolic software is used. In the current paper, the software package `GeM` [24] for `Maple` has been used.

4.1. *Point transformations and point symmetries*

Consider a PDE system

$$E^{\sigma}(z,u,\partial u,\ldots,\partial^k u)=0,\qquad \sigma=1,\ldots,N, \tag{25}$$

involving derivatives of order at most k, with n independent variables $z=(z^1,\ldots,z^n)$ and m dependent variables $u(z)=(u^1(z),\ldots,u^m(z))$.

A *one-parameter Lie group of point transformations* of variables (z,u) is a one-to-one transformation acting on the $n+m$-dimensional space (z,u) of the form

$$z^*=f(z,u;\varepsilon),\qquad u^*=g(z,u;\varepsilon) \tag{26}$$

where functions f, g are such that transformations (26) have a structure of a Lie group with the parameter ε. Eq. (26) define the global form of the point transformation group; the corresponding local form of the transformations (26) is given by

$$\begin{aligned}(z^*)^i&=f^i(z,u;\varepsilon)=z^i+\varepsilon\xi^i(z,u)+O(\varepsilon^2),\quad i=1,\ldots,n,\\ (u^*)^{\mu}&=g^{\mu}(z,u;\varepsilon)=u^{\mu}+\varepsilon\eta^{\mu}(z,u)+O(\varepsilon^2),\quad \mu=1,\ldots,m.\end{aligned} \tag{27}$$

When the independent variables z and the dependent variables $u(z)$ are transformed, the partial derivatives of $u(z)$ are transformed in a prescribed way; the corresponding formulas can be found in any standard text, e.g., [3]. Such formulas are called *prolongation formulas*.

A one-parameter Lie group of point transformations (26) is *a group of point symmetries* of a given PDE system if and only if its corresponding prolongation leaves invariant the solution manifold of a given PDE system in the space that includes all its independent variables z, dependent variables $u(z)$, and all necessary partial derivatives of $u(z)$. In other words, a point symmetry is a point transformation (26) such that if $u(z)$ is a solution of the given PDE system (25), then $u^*(z^*)$ is also a solution of the same system.

In order to find a point symmetry (26), it is necessary and sufficient to find all functions $\xi^i(z,u)$, $\eta^{\mu}(z,u)$ entering (27). These quantities are commonly referred to as components of an *infinitesimal generator*

$$\mathrm{Y}=\xi^i(z,u)\frac{\partial}{\partial x^i}+\eta^{\mu}(z,u)\frac{\partial}{\partial u^{\mu}}. \tag{28}$$

Symmetry components $\xi^i(z,u)$, $\eta^{\mu}(z,u)$ are found in an algorithmic fashion from *determining equations* that involve the application of the prolongation $\mathrm{Y}^{(k)}$ of the infinitesimal generator (28) to equations of a given system:

$$\mathrm{Y}^{(k)}E^{\alpha}(z,u,\partial u,\ldots,\partial^k u)=0,\qquad \alpha=1,\ldots,N, \tag{29}$$

on solutions of (25). Details on formulas for $\mathrm{Y}^{(k)}$ and other aspects of computation of local symmetries are found in [3] or any standard symmetry text.

4.2. Equivalence transformations

The notion of equivalence transformations is closely related to the notion of point symmetries. Equivalence transformations preserve the differential structure of equations but *change the constitutive functions and/or parameters* of the model. Equivalence transformations are used to narrow down the set of constitutive functions/parameters that need to be considered separately. This is important, for example, in analyses that involve classifications.

This notion has raised some interest amongst researchers, and equivalence groups have been computed by a few authors [32–34] however restricting to balance equations written in generic form without consideration for general nonlinear elasticity problems. We presently advocate a novel contribution by exposing a general methodology for calculating equivalence transformations of 2D hyperelastic models; we shall restrict to a few constitutive models amongst the type presented in Table 2.

Consider a family of PDE systems (25) involving a set of constitutive functions and/or parameters $K = (K_1, \ldots, K_L)$. Then *a one-parameter Lie group of equivalence transformations* is a one-parameter Lie group of transformations given by

$$\begin{aligned} \tilde{z}^i &= f^i(z, u; \varepsilon), \quad i = 1, \ldots, n, \\ \tilde{u}^\mu &= g^\mu(z, u; \varepsilon), \quad \mu = 1, \ldots, m, \\ \tilde{K}_l &= G_l(z, u, K; \varepsilon), \quad l = 1, \ldots, L, \end{aligned} \tag{30}$$

which maps a given PDE system with constitutive functions/parameters K into another one in the family, with a new set of constitutive functions/parameters $\tilde{K}$.

Equivalence transformations can be computed, similarly to symmetries, through solutions of determining equations of the form (29), with a restriction that variables on which symmetry components corresponding to constitutive functions/parameters depend are appropriately specified. For example, if $K_1 = a$ and $K_2 = b$ are two constant constitutive parameters of the Mooney–Rivlin model (15), the corresponding equivalence transformations (for these parameters) will be of the form

$$\tilde{a} = G_1(a, b; \varepsilon), \qquad \tilde{b} = G_2(a, b; \varepsilon),$$

which does not involve any Eulerian and Lagrangian coordinates or time, since $a, b = \text{const}$. Similarly, suppose that some model involves a constitutive function $K_1 = Q(y)$ where y is an independent variable. Then the corresponding equivalence transformations for $Q(y)$ will be of the form

$$\tilde{Q}(\tilde{y}) = G_1(y, Q(y); \varepsilon);$$

this transformation will not involve other variables than x, since $Q = Q(y)$ and $\tilde{Q} = \tilde{Q}(\tilde{y})$ only. For further details, see, e.g., [3].

5. Equivalence transformations and a simplified constitutive relation for 2D Ciarlet–Mooney–Rivlin models

Starting at this point, we restrict attention to purely two-dimensional motions with $x^{1,2} = x^{1,2}(X^1, X^2, t)$; the third coordinate $x^3 = X^3$ is fixed. The deformation gradient matrix is given by

$$\mathbf{F} = \begin{bmatrix} F_{11} & F_{12} & 0 \\ F_{21} & F_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The equations of motion (5) correspondingly reduce to two nonlinear dynamic PDEs

$$\begin{aligned} \rho_0 (x^1)_{tt} - \frac{\partial P^{11}}{\partial X^1} - \frac{\partial P^{12}}{\partial X^2} - \rho_0 R^1 = 0, \\ \rho_0 (x^2)_{tt} - \frac{\partial P^{21}}{\partial X^1} - \frac{\partial P^{22}}{\partial X^2} - \rho_0 R^2 = 0, \end{aligned} \tag{31}$$

where

$$\rho_0 = \rho_0(X^1, X^2), \qquad P^{ij} = \rho_0(X^1, X^2) \frac{\partial W}{\partial F_{ij}}, \quad i, j = 1, 2, \tag{32}$$

and the Ciarlet–Mooney–Rivlin strain energy function is given by a general expression (16).

Eq. (31) are nonlinear; they become linear only in the neo-Hookean case when $a > 0, b = c = d = 0$, which however admits no natural state and therefore is non-physical (Section 3).

Seeking equivalence transformations that preserve the structure of Eq. (31) and treating constants a, b, c, d as constitutive parameters and $\rho_0(X^1, X^2)$ and $R^i(X^1, X^2, t), i = 1, 2$, as constitutive functions, we arrive at the following results.

Theorem 3. *The two-dimensional elastodynamics equations (31) and (32) with the Ciarlet–Mooney–Rivlin constitutive law (16) admit the equivalence transformations given by formulas*

$$\begin{aligned}
&\tilde{t} = e^{\varepsilon_2} t + \varepsilon_1,\\
&\tilde{X}^1 = e^{\varepsilon_3}\left(X^1 \cos\varepsilon_7 - X^2 \sin\varepsilon_7\right) + \varepsilon_4,\\
&\tilde{X}^2 = e^{\varepsilon_3}\left(X^1 \sin\varepsilon_7 + X^2 \sin\varepsilon_7\right) + \varepsilon_5,\\
&\tilde{x}^1 = e^{2\varepsilon_2} x^1 + f^1(t),\\
&\tilde{x}^2 = e^{2\varepsilon_2} x^2 + f^2(t),\\
&\tilde{\rho}_0 = e^{\varepsilon_6}\rho_0,\\
&\tilde{R}^1 = R^1 + \frac{d^2 f^1(t)}{dt^2}, \qquad \tilde{R}^2 = R^2 + \frac{d^2 f^2(t)}{dt^2},\\
&\tilde{a} = -b + e^{2\varepsilon_3 - 2\varepsilon_2}(a+b), \qquad \tilde{b} = b,\\
&\tilde{c} = -b + e^{4\varepsilon_3 - 6\varepsilon_2}(b+c), \qquad \tilde{d} = e^{2\varepsilon_2} d,
\end{aligned} \tag{33}$$

and an additional equivalence transformation given in the local form by the formula

$$\begin{aligned}
&\tilde{a} = a + \varepsilon_8 G(a,b,c,d) + O(\varepsilon^2),\\
&\tilde{b} = b - \varepsilon_8 G(a,b,c,d) + O(\varepsilon^2),\\
&\tilde{c} = c + \varepsilon_8 G(a,b,c,d) + O(\varepsilon^2),\\
&\tilde{d} = d.
\end{aligned} \tag{34}$$

In (33) and (34), $\varepsilon_1, \ldots \varepsilon_8$ are arbitrary constants, and $f^1(t), f^2(t)$, $G(a,b,c,d)$ are arbitrary functions of their arguments.

Proof. The proof follows from a direct computation of equivalence transformations of the PDE system (31) with (16) and (32) as discussed in Section 4. □

The equivalence transformations (33) and (34) include scalings, translations and rotations of the material coordinates (parameters $\varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_7$), Galilean transformations and translations of Eulerian coordinates and time (parameters $\varepsilon_1, \varepsilon_2$, arbitrary functions $f_1(t), f_2(t)$), scaling of the body density in the reference configuration (parameter ε_6), and most importantly, transformations of parameters a, b, c, d of the constitutive model (parameters $\varepsilon_2, \varepsilon_3$, and the transformation (34)). The latter, after some computation, yield the following important result.

Theorem 4 (*Principal Result 1*)**.** *The dynamics of Ciarlet–Mooney–Rivlin models (16) in two dimensions depends only on three constitutive parameters. In particular, the two-dimensional first Piola–Kirchhoff stress tensor can be written as*

$$\mathbf{P_2} = \rho_0\left[A\mathbf{F_2} + BJ\mathbf{C_2} - \frac{d}{J}\mathbf{C_2}\right], \tag{35}$$

where $A = 2(a+b) \geq 0$, $B = 2(b+c) \geq 0$ and d are the three independent constitutive parameters of the model, and

$$\mathbf{F_2} = \begin{bmatrix} F^1_{\ 1} & F^1_{\ 2} \\ F^2_{\ 1} & F^2_{\ 2} \end{bmatrix}, \qquad \mathbf{C_2} = \begin{bmatrix} F^2_{\ 2} & -F^2_{\ 1} \\ -F^1_{\ 2} & F^1_{\ 1} \end{bmatrix}$$

are respectively the two-dimensional deformation gradient and its cofactor matrix.

Proof. First we show that for Ciarlet–Mooney–Rivlin constitutive models (16) with arbitrary constant parameters a, b, c, d, the equations of motion (31) depend on only three independent parameters. This follows from the equivalence transformations (33) for a and c. First, if $b = 0$, the model includes only three constitutive parameters a, c, d. Second, if $b \neq 0$, one can choose the group parameter values

$$\varepsilon_3 = \frac{1}{4}\ln\left(\frac{b}{b+c}\right), \qquad \varepsilon_i = 0, \quad i = 1, 2, 4, \ldots, 8$$

to obtain the transformed constitutive parameters

$$\tilde{a} = -b + (a+b)\sqrt{\frac{b}{b+c}}, \qquad \tilde{b} = b, \qquad \tilde{c} = 0, \qquad \tilde{d} = d.$$

Since $\tilde{c} = 0$, the model depends on three constitutive parameters $\tilde{a}$, $\tilde{b}$ and $\tilde{d}$. Observing from (33) that

$$\tilde{a} + \tilde{b} = e^{2\varepsilon_3 - 2\varepsilon_2}(a+b), \qquad \tilde{b} + \tilde{c} = e^{4\varepsilon_3 - 6\varepsilon_2}(b+c),$$

Table 3
Point symmetry classification for the two-dimensional Ciarlet–Mooney–Rivlin models (31), (35) and (37) with zero forcing and $\rho_0 = \text{const} > 0$.

Case	*Point symmetries*
General	$Y_1 = \frac{\partial}{\partial t}, Y_2 = \frac{\partial}{\partial X^1}, Y_3 = \frac{\partial}{\partial X^2}, Y_4 = \frac{\partial}{\partial x^1}, Y_5 = \frac{\partial}{\partial x^2}, Y_6 = t\frac{\partial}{\partial x^1}, Y_7 = t\frac{\partial}{\partial x^2}, Y_8 = X^2\frac{\partial}{\partial X^1} - X^1\frac{\partial}{\partial X^2}, Y_9 = x^2\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^2}, Y_{10} = t\frac{\partial}{\partial t} + X^1\frac{\partial}{\partial X^1} + X^2\frac{\partial}{\partial X^2} + x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2}$
$A = 0$; B, d arbitrary	$Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7, Y_8, Y_9, Y_{10}, Y_{11} = f_1(X^2)\frac{\partial}{\partial x^1}, Y_{12} = \left(\frac{\partial}{\partial X_2} f_2(X^1, X^2)\right)\frac{\partial}{\partial x^1} - \left(\frac{\partial}{\partial X_1} f_2(X^1, X^2)\right)\frac{\partial}{\partial x^2}$; $f_1(X^2), f_2(X^1, X^2)$ are arbitrary functions
$A = d = 0$; B arbitrary	$Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7, Y_8, Y_9, Y_{10}, Y_{11}, Y_{12}, Y_{13} = t\frac{\partial}{\partial t} + X^1\frac{\partial}{\partial X^1}$
$A = B = 0$; d arbitrary	$Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7, Y_8, Y_9, Y_{10}, Y_{11}, Y_{12}, Y_{14} = X^1\frac{\partial}{\partial X^1}$

one can denote $A = 2(a + b)$, $B = 2(b + c)$, and rewrite equivalence transformations (33) in a more compact form

$$\begin{aligned}
&\widetilde{t} = e^{\varepsilon_2} t + \varepsilon_1,\\
&\widetilde{X}^1 = e^{\varepsilon_3}\left(X^1\cos\varepsilon_7 - X^2\sin\varepsilon_7\right) + \varepsilon_4,\\
&\widetilde{X}^2 = e^{\varepsilon_3}\left(X^1\sin\varepsilon_7 + X^2\sin\varepsilon_7\right) + \varepsilon_5,\\
&\widetilde{x}^1 = e^{2\varepsilon_2}x^1 + f^1(t),\\
&\widetilde{x}^2 = e^{2\varepsilon_2}x^2 + f^2(t),\\
&\widetilde{\rho_0} = e^{\varepsilon_6}\rho_0,\\
&\widetilde{R}^1 = R^1 + \frac{d^2 f^1(t)}{dt^2}, \qquad \widetilde{R}^2 = R^2 + \frac{d^2 f^2(t)}{dt^2},\\
&\widetilde{A} = e^{2\varepsilon_3 - 2\varepsilon_2}A, \qquad \widetilde{B} = e^{4\varepsilon_3 - 6\varepsilon_2}B,\\
&\widetilde{d} = e^{2\varepsilon_2}d.
\end{aligned} \tag{36}$$

The form of the two-dimensional first Piola–Kirchhoff stress tensor (35) is obtained by a straightforward componentwise computation using the constitutive law (16) and the formula (32). □

In Sections 6 and 7, symmetry classification of two-dimensional Ciarlet–Mooney–Rivlin dynamic equations are given in terms of the essential parameters A, B, d modulo the equivalence transformations (36).

6. Symmetry classification for time-dependent 2D Ciarlet–Mooney–Rivlin models

In the current section, we classify symmetries of the two-dimensional Ciarlet–Mooney–Rivlin models given by (31) and (35) with respect to the values of constants $A, B, d \geq 0$, and types of functions $\rho_0(X^1, X^2)$ that appear in the equations. We restrict to the important case of zero external body forces:

$$R^1(t, X^1, X^2) = R^1(t, X^1, X^2) = 0, \qquad \rho_0(X^1, X^2) = \text{const}, \tag{37}$$

which is naturally expected to have a richer symmetry structure. The following theorem holds.

Theorem 5. *Modulo the equivalence transformations (36), the full symmetry classification of the PDE system (31), (35) and (37) is given in Table 3.*

Proof. The proof of Theorem 5 proceeds by direct computation. □

In Table 3, symmetry generators Y_1, Y_2, Y_3, Y_4, Y_5 correspond to translations in dependent and independent variables; generators Y_6, Y_7 correspond to Galilean transformations

$$x^1 \to x^1 + \varepsilon t, \quad x^2 \to x^2 + \varepsilon t;$$

generators Y_8, Y_9 correspond to rotations in material and Eulerian coordinates respectively; generators Y_{10}, Y_{13}, Y_{14} correspond to scaling transformations.

Each of the generators Y_{11}, Y_{12} corresponds to an infinite number of symmetries through an arbitrary function.

7. Classification of symmetries of the 2D Ciarlet–Mooney–Rivlin models in traveling wave coordinates

Only a few authors have considered the propagation of elastic waves in compressible solids; examples include [20] and the recent reference [26].

We now consider the two-dimensional Ciarlet–Mooney–Rivlin models given by (31), in traveling wave coordinates, assuming that the medium moves with respect to the observer with a constant speed $s > 0$ in some prescribed direction. Without loss of generality, one may choose that direction to be along the axis of X^1. In this ansatz, one has

$$x^i(X^1, X^2, t) = w^i(z, X^2), \qquad z = X^1 - st, \quad i = 1, 2. \tag{38}$$

Table 4
Point symmetry classification for the two-dimensional Ciarlet–Mooney–Rivlin models in traveling-wave coordinates, given by (31), (35) and (39), with zero forcing and $\rho_0 = \rho_0(X^2)$.

Case number	Case	*Point symmetries*
1	General	$Y_1 = \frac{\partial}{\partial z}, Y_2 = \frac{\partial}{\partial w^1}, Y_3 = \frac{\partial}{\partial w^2}, Y_4 = w^2 \frac{\partial}{\partial w^1} - w^1 \frac{\partial}{\partial w^2}$
2	$\rho_0(X^2) = (X^2 + q_1)^{q_2}$; $q_1, q_2 =$ const, $q_2 \neq 0$; A, B, d, s arbitrary	$Y_1, Y_2, Y_3, Y_4, Y_5 = z\frac{\partial}{\partial z} + (X^2 + q_1)\frac{\partial}{\partial X^2} + w^1 \frac{\partial}{\partial w^1} + w^2 \frac{\partial}{\partial w^2}$
3a	$\rho_0(X^2) = \exp(q_1 X^2)$; $q_1 =$ const $\neq 0$; A, B, d, s arbitrary	$Y_1, Y_2, Y_3, Y_4, Y_6 = \frac{\partial}{\partial X^2}$
3b	$\rho_0(X^2) = \exp(q_1 X^2)$; $q_1 =$ const $\neq 0$; A, d arbitrary; $B = 0$; $\boxed{s^2 = A}$	$Y_1, Y_2, Y_3, Y_4, Y^{\infty}_{(1)} = -\left(\frac{1}{q_1}\frac{d}{dz} f_1(z)\right)\frac{\partial}{\partial X^2} + f_1(z)\frac{\partial}{\partial z}$; f1(z) is an arbitrary function
4a	$\rho_0(X^2) > 0$ arbitrary; A, B arbitrary; $d = 0$; $\boxed{s^2 = A}$	$Y_1, Y_2, Y_3, Y_4, Y_7 = z\frac{\partial}{\partial z} + w^1 \frac{\partial}{\partial w^1} + w^2 \frac{\partial}{\partial w^2}, Y_8 = \left(\rho_0 \int \frac{1}{\rho_0} dX^2\right)\frac{\partial}{\partial X^2}, Y^{\infty}_{(2)} = f_2(z)\rho_0 \frac{\partial}{\partial X^2}$; $f_2(z)$ is an arbitrary function
4b	$\rho_0(X^2) > 0$ arbitrary; A, d arbitrary; $B = 0$; $\boxed{s^2 = A}$	$Y_1, Y_2, Y_3, Y_4, Y_9 = z\frac{\partial}{\partial z}$
5a	$\rho_0 =$ const; A, B, d, s arbitrary	$Y_1, Y_2, Y_3, Y_4, Y_5(q_1 = 0), Y_6, Y_{10} = X^2 \frac{\partial}{\partial z} - \frac{Az}{A - s^2}\frac{\partial}{\partial X^2}$
5b	$\rho_0 =$ const; $\boxed{s^2 = A}$; A, B, d arbitrary	$Y_1, Y_2, Y_3, Y_4, Y_5(q_1 = 0), Y^{\infty}_{(3)} = f_3(z)\frac{\partial}{\partial X^2}$; $f_3(z)$ is an arbitrary function
5c	$\rho_0 =$ const; $\boxed{s^2 = A}$; A, d arbitrary; $B = 0$	$Y_1, Y_2, Y_3, Y_4, Y_5(q_1 = 0), Y_9, Y^{\infty}_{(3)}$

In such a setting, for a symmetry classification, it is natural to consider the body density in the reference configuration in the form

$$\rho_0 = \rho_0(X^2), \tag{39a}$$

and the no-forcing case

$$R^1(X^1, X^2, t) = R^2(X^1, X^2, t) = 0. \tag{39b}$$

The difference between the current case and the full dynamic case of Section 6 is the form of the derivatives in PDEs (31) and (35):

$$\frac{\partial^2}{\partial t^2} x^i(X^1 - st, X^2) = -s^2 \frac{\partial^2}{\partial z^2} w^i(z, X^2), \quad i = 1, 2.$$

The following result summarizes the symmetry classification.

Theorem 6 (*Principal Result 2*)**.** *Modulo the equivalence transformations (36), the full symmetry classification of the PDE system (31), (35) and (39) in traveling wave coordinates (38) is given in Table 4.*

In particular, the general non-linear Ciarlet–Mooney–Rivlin model possesses a specific traveling wave speed, for which the equations admit additional symmetries, including additional infinite set of point symmetries given by generators $Y^{\infty}_{(1)}, Y^{\infty}_{(2)}, Y^{\infty}_{(3)}$. This special wave speed is given by

$$s = s_* = \sqrt{A} = \sqrt{2(a + b)} \tag{40}$$

and is equal to the constant wave speed of the linear (neo-Hookean, $B = d = 0$) version of the dynamic equations (31) and (35).

Proof. The part of the proof related to symmetry classification proceeds by direct computations. The proof of the last statement of the theorem is established as follows. In the neo-Hookean case $W = AI_1$, $\mathbf{P_2} = \rho_0 A \mathbf{F_2}$, $\rho_0 =$ const, and the two-dimensional elastodynamic equations (31) and (35), become a decoupled system of linear wave equations

$$\begin{aligned} (x^1)_{tt} &= A\left(\frac{\partial^2 x^1}{\partial (X^1)^2} + \frac{\partial^2 x^1}{\partial (X^2)^2}\right), \\ (x^2)_{tt} &= A\left(\frac{\partial^2 x^2}{\partial (X^1)^2} + \frac{\partial^2 x^2}{\partial (X^2)^2}\right), \end{aligned} \tag{41}$$

which has the characteristic wave speed $s_{\text{lin}} = \sqrt{A}$, which coincides with the special wave speed (40) that arises for the full nonlinear Ciarlet–Mooney–Rivlin model: $s_* = s_{\text{lin}}$. □

In Table 4, symmetry generators Y_1, Y_2, Y_3 correspond to translations in dependent and independent variables; generator Y_4 corresponds to rotations in Eulerian coordinates; generators Y_5, Y_7, Y_9, Y_{10} correspond to scaling transformations. Additional symmetries Y_7, Y_8, Y_9, Y_{10} and additional infinite families of symmetries $Y^{\infty}_{(1)}, Y^{\infty}_{(2)}, Y^{\infty}_{(3)}$ arise for the special value (40) of the translation speed.

Remark 4. The statement of Theorem 6 may be also interpreted backwards. If the Ciarlet–Mooney–Rivlin constitutive relation (16) is believed to be correct for a given model, and wave propagation with wave speed s_* is observed, then the relation

$$2(a+b)=s_*^2$$

could be used to predict the value of the parameter b from the known parameter a or vice versa. This conjecture however requires experimental verification.

An example

The full study of invariant solutions arising from symmetries listed in Table 4 generally leads to highly complicated ordinary differential equations, and is left to future work. We currently present a specific example of exact solutions of the two-dimensional elastodynamics equations (31) and (32) with the Ciarlet–Mooney–Rivlin constitutive law (16) in traveling wave coordinates (38), obtained using a symmetry from Table 4 and equivalence transformations (36).

Consider solutions corresponding to the special wave speed (40), in the case of constant body density and no body forces, in a setting corresponding to a natural state (cf. (24)). One has

$$\rho_0=\text{const},\qquad R^1=R^2=0,\qquad A=s^2,\qquad d=s^2+B. \tag{42}$$

Without loss of generality, $s>0$. A basic solution

$$w^1=z \Leftrightarrow x^1(X^1,X^2,t)=X^1-st,\qquad w^2=x^2(X^1,X^2,t)=X^2, \tag{43}$$

describes an undistorted infinite two-dimensional medium moving past the observer to the left (i.e., in the negative X^1 direction) with the speed s. Since the setup falls in the case 5b of Table 4, one may use the symmetry $\mathrm{Y}^{\infty}_{(3)}$, which maps the solution (43) into a family of exact solutions

$$\begin{aligned} &w^1=z \Leftrightarrow x^1(X^1,X^2,t)=X^1-st,\\ &w^2=X^2-f(z) \Leftrightarrow x^2(X^1,X^2,t)=X^2-f(X^1-st) \end{aligned} \tag{44}$$

of the same model, holding for all values of ρ_0, B, and s. Solutions (44) involve an arbitrary function $f(z)$; by design, they are static in the moving frame of reference, and correspond to right-propagating deformations in an infinite two-dimensional elastic medium. The corresponding first Piola–Kirchhoff stress tensor for solutions (44) is given by

$$\mathbf{P_2}=-\rho_0 s^2 f'(X^1-st)\begin{bmatrix}0&1\\1&0\end{bmatrix}. \tag{45}$$

In order to display solutions (44), consider a rectangular grid of lines $X^1=\text{const}$ and $X^2=\text{const}$ that labels material particles in the reference (Lagrangian) frame. The corresponding parametric curves given by $(x^1(X^1,X^2,t),x^2(X^1,X^2,t))$ for $X^1=\text{const}$ or $X^2=\text{const}$ show the "distorted" grid corresponding to actual positions of the labeled material particles at any given time $t=\text{const}$. In the coordinate frame of an observer moving with speed s (to the right when $s>0$), the solution profile is time-independent; the deformed grid is obtained by plotting the parametric curves $(w^1(z,X^2),w^2(z,X^2))$ at $z=\text{const}$ (vertical direction) and $X^2=\text{const}$ (horizontal direction). Sample graphs are given in Fig. 2.

One may further use equivalence transformations (36) in order to get, for example, scaled or rotated versions of solutions (44), and/or solutions corresponding to waves traveling with a different speed s.

For example, consider a traveling wave-type exact solution of the type (44) in an elastic medium with prescribed constitutive parameters A^*, B^*, d^*, propagating with speed $s^*=\sqrt{A^*}$:

$$x^1(X^1,X^2,t)=X^1-st,\qquad x^2(X^1,X^2,t)=X^2+\alpha\exp(-\beta(X^1-st)^2), \tag{46}$$

where α,β are some fixed constants of appropriate physical dimensions. Using equivalence transformations (36) with parameters

$$\varepsilon_2=-\frac{1}{2}\ln p,\qquad \varepsilon_3=-\frac{1}{2}\ln q,\quad p,q>0,\qquad \varepsilon_1=\varepsilon_4=\cdots=\varepsilon_8=0,$$

one arrives at an exact solution

$$\begin{aligned} &\tilde{x}^1(\tilde{X}^1,\tilde{X}^2,\tilde{t})=\frac{\sqrt{q}}{p}\left(\tilde{X}^1-s\sqrt{\frac{p}{q}}\,\tilde{t}\right),\\ &\tilde{x}^2(\tilde{X}^1,\tilde{X}^2,\tilde{t})=\frac{\sqrt{q}}{p}\left(\tilde{X}^2+\alpha\exp\left[-\beta\sqrt{q}\left(\tilde{X}^1-s\sqrt{\frac{p}{q}}\,\tilde{t}\right)\right]^2\right). \end{aligned} \tag{47}$$

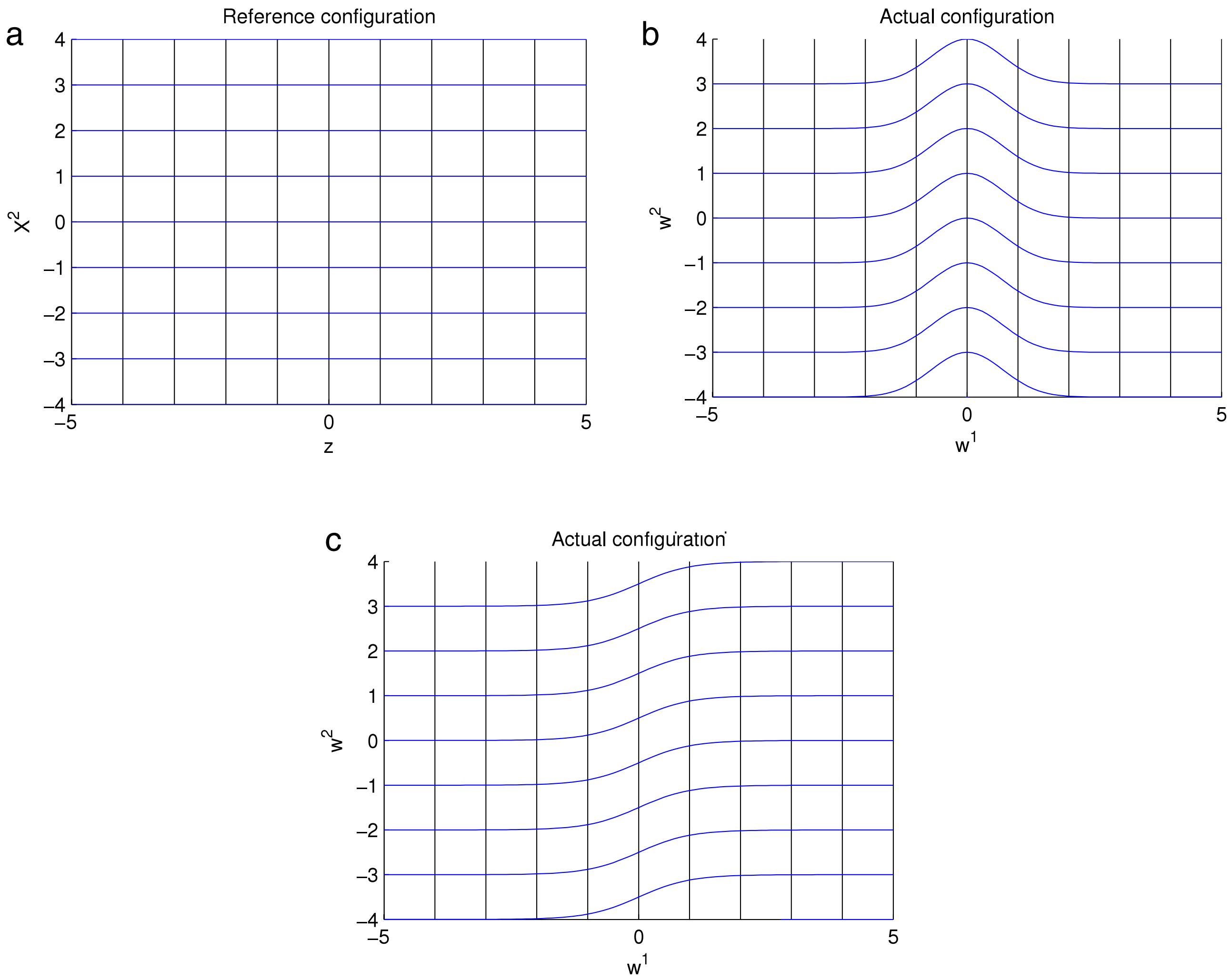

Fig. 2. (a) A rectangular grid in the reference configuration. (b) and (c) The deformation corresponding to the exact solutions (44) in the actual (Euler) configuration, in the frame of the observer traveling with speed s, for the cases $f(z) = -\exp(-z^2)$ and $f(z) = -(1 + \tanh z)/2$, respectively.

The solution (47) propagates with the speed

$$\tilde{s} = s\sqrt{\frac{p}{q}},$$

and corresponds to an elastic material with a different set of constitutive parameters, given by

$$\tilde{A} = \frac{p}{q}A^*, \qquad \tilde{B} = \frac{p^3}{q^2}B^*, \qquad \tilde{d} = p^{-1}d^*.$$

It is important to note that the above family of exact solutions corresponds to essentially nonlinear waves, and generally holds only for the indicated Ciarlet–Mooney–Rivlin model.

Due to the highly complicated explicit form of the dynamic equations (31) and their linearized versions, application of results like [35] to stability analysis of the exact solutions of the form (44) presents a significant technical difficulty. We expect that stability properties of solutions (44) will depend on the form of the arbitrary function $f(z)$. It is planned to address this question in future research.

8. Conclusions

In the current paper, isotropic homogeneous hyperelastic materials whose dynamics is described by equations of motion (5) with a constitutive law of the type (13) were studied. In Section 2, details on equations of motion and some common constitutive relations were presented.

In Section 3, the dependence between a constitutive law and the existence of a natural state was discussed, i.e., whether or not, for a given constitutive law, zero displacement implies zero stress. This question is of high practical importance, for example, for consistent formulation of boundary value problems for numerical computations. Theorem 2 gives an explicit condition for the form of the strain energy density in terms of invariants. In particular, it follows that the classical neo-Hookean and Mooney–Rivlin models (15) do not correspond to a natural state, whereas Ciarlet–Mooney–Rivlin materials (16) and, more generally, Hadamard materials (14) may admit a natural state, depending on the parameters used.

The paper subsequently concentrates on two-dimensional planar motions of Ciarlet–Mooney–Rivlin elastic solids. In Section 5, equivalence transformations of that model were computed, and it was shown that in fact, the equations of motion only depend on three independent quantities, which are combinations of four standard parameters a, b, c, d of the Ciarlet–Mooney–Rivlin model (16). An explicit simple form of the two-dimensional first Piola–Kirchhoff stress tensor (35) was derived.

Further, point symmetries of the two-dimensional Ciarlet–Mooney–Rivlin equations were classified, using symbolic software, in the full dynamic setting (Section 6). The knowledge of these symmetries is important for the ongoing work of computation of exact solutions and conservation laws of the Ciarlet–Mooney–Rivlin model. In particular, for potential or no forcing, the conservation law and symmetry structures are related through Noether's theorem, since the governing equations admit a variational formulation (see the Appendix).

In Section 7, point symmetries of the two-dimensional Ciarlet–Mooney–Rivlin model in traveling wave coordinates were classified. A special wave speed value was found for which the Ciarlet–Mooney–Rivlin model admits additional symmetries, in particular, infinite additional symmetries in the case of the constant body density. A family of exact solutions corresponding to nonlinear two-dimensional elastic waves traveling in an infinite medium was presented.

Future research is expected to include the following directions.

- Application of the computed symmetries to the construction of further exact solutions and conservation laws of the considered elasticity models, and to the development of specialized numerical schemes that are based on these admitted symmetries and conservation laws.
- Computation of symmetry structure of elasticity models in the important non-planar two-dimensional reductions (including axial symmetry), and in three dimensions.
- Generalization to other constitutive models. In particular, it is planned to consider in detail the case of anisotropic media consisting of a hyperelastic matrix reinforced by elastic fibers.

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Appendix. Lagrangian and Hamiltonian formulations

It can be shown that the Eqs. (5a) of conservation of momentum follow from a variational principle [29]. Suppose that external forces $\mathbf{R}$ are potential, with potential energy $\mathcal{V}_R(\mathbf{x})$:

$$\mathbf{R} = -\nabla \mathcal{V}_R(\mathbf{x}).$$

Define total potential and kinetic energy of the configuration occupying reference volume $\overline{\Omega}_0$ by

$$V(\mathbf{x}) = \int_{\Omega_0} \rho_0 (W(\mathbf{F}) + \mathcal{V}_R(\mathbf{x})) dV, \qquad K(\mathbf{u}) = \frac{1}{2} \int_{\Omega_0} \rho_0 \|\mathbf{x}_t\|^2 dV. \tag{48}$$

The Hamiltonian and the Lagrangian are defined by

$$H = K + V, \qquad L = K - V = \int_{\Omega_0} \mathfrak{L}(\mathbf{x}, \mathbf{x}_t, \mathbf{F}) dV \tag{49}$$

where the Lagrangian density is given by

$$\mathfrak{L}(\mathbf{x}, \mathbf{x}_t, \mathbf{F}) = \frac{1}{2} \rho_0 \|\mathbf{x}_t\|^2 - \rho_0 (W(\mathbf{F}) + \mathcal{V}_R(\mathbf{x})).$$

When variables involved are sufficiently smooth, extremizing the action yields the strong Euler–Lagrange equations

$$\frac{\partial}{\partial t} D_{\mathbf{x}_t} \mathfrak{L}(\mathbf{x}, \mathbf{x}_t, \mathbf{F}) = D_{\mathbf{x}} \mathfrak{L}(\mathbf{x}, \mathbf{x}_t, \mathbf{F}), \tag{50}$$

which are equivalent to the equations of motion (5a).

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