Applications of Symmetry Analysis in Stability Theory

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Summary

For decades the stability of nearly parallel shear flows was primarily analyzed employing the Orr-Sommerfeld-Equation (OSE). We show that the OSE is solely based on three symmetries of the linearized Navier-Stokes-Equation for two-dimensional perturbations. In fact, the OSE is a similarity reduction using the latter three symmetries. Though rather successful in boundary layer flows the OSE does not give proper results for the plane channel flow with the classical parabolic flow profile. For this special case we found a new symmetry. It leads to a new ansatz considerably distinct from the OSE with two new similarity variables. We analyzed the scope in which the new ansatz could be used. Finally, we derived a technique to solve the equation via the new ansatz function.

1 Introduction

We consider a parallel base flow $(U(y), 0, 0)^T$ with a two-dimensional perturbation of the form $(u(x, y), v(x, y), 0)^T$. Assume that the Navier-Stokes-Equation holds for the base flow. For the perturbated flow, we get the following set of equations:

$$\frac{\partial u}{\partial t} + U(y) \cdot \frac{\partial u}{\partial x} + v \cdot \frac{dU}{dy} + \left\{ u \cdot \frac{\partial u}{\partial x} + v \cdot \frac{\partial u}{\partial y} \right\} = -\frac{1}{\varrho} \cdot \frac{\partial p}{\partial x} + \nu \Delta u, \tag{1}$$

$$\frac{\partial v}{\partial t} + U(y) \cdot \frac{\partial v}{\partial x} + \left\{ u \cdot \frac{\partial v}{\partial x} + v \cdot \frac{\partial v}{\partial y} \right\} = -\frac{1}{\varrho} \cdot \frac{\partial p}{\partial y} + \nu \Delta v, \tag{2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{3}$$

Introducing a stream function ψ with $u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$ eliminates the continuity equation. Finally, applying the curl gives us equation (4).

$$\frac{\partial}{\partial t} \Delta \psi - \frac{d^2 U}{dy^2} \cdot \frac{\partial \psi}{\partial x} + U \cdot \frac{\partial}{\partial x} \Delta \psi + \nabla \times (\nabla \psi \cdot \Delta \psi) = \nu \Delta \Delta \psi \tag{4}$$

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Mostly we will consider the linearized equation for our further considerations.

$$\frac{\partial}{\partial t}\Delta\psi - \frac{d^2U}{dy^2}\cdot\frac{\partial\psi}{\partial x} + U\cdot\frac{\partial}{\partial x}\Delta\psi = \nu\Delta\Delta\psi \tag{5}$$

We will show how equation (5) systematically leads to the Orr-Sommerfeld-Equation using symmetry analysis. We found one additional symmetry for the parabolic channel flow. This extended set of symmetries leads to a new ansatz function different from the normal mode ansatz. The structure of the new ansatz function has some properties which could be of great interest for the stability theory of channel flows.

2 Symmetry Analysis

Symmetries are an important tool to analyze differential equations. They map the solution manifold of a DE into itself. One point symmetries can be found by means of the Lie-Algorithm [4]. With the help of symmetry analysis, solutions of DE can be found, e.g. using invariant solutions. We say that a solution of a DE is invariant with respect to a symmetry if it is mapped onto itself by this symmetry. For ordinary differential equations, the general solution can be found using symmetry analysis. In the case of partial differential equations we may for the most cases only expect to find special solutions [1, 4].

In the field of fluid mechanics experience has shown that many flows turn out to be invariant under certain symmetries. Also many classical ansatz functions turn out to be invariant under special symmetries. In particular we will show that this is the case of the ansatz leading to the Orr-Sommerfeld-Equation. The present symmetry analysis was done using the GeM package of Cheviakov [3] and the DESOLVE package of Carminati and Vu [2].

2.1 Symmetries Leading to the Orr-Sommerfeld-Equation

For general flow profiles $U(y)$, equation (5) has three symmetries with the following infinitesimals:

$$X_0 = f(x,y,t)\partial_\psi \tag{6}$$

$$X_1 = \partial_x \tag{7}$$

$$X_2 = \partial_t \tag{8}$$

$$X_3 = \psi\partial_\psi \tag{9}$$

(7) and (8) correspond to translation invariances in x-direction and time. (9) is a scaling symmetry in ψ which is a consequence of the linearity of equation (5). The symmetry (6) refers to the superposition principle of any linear DE since f is a solution of the original equation under investigation, here (5). Note that this is also always implied in the OSE. Still it will be always employed subsequently without explicitly mention any more. We formulate a general symmetry as a combination of (7)-(9): $X := \alpha X_1 + X_2 + \gamma X_3$ with $\alpha, \gamma \in \mathbb{C}$. The respective invariant solution is given by

$$\psi = f(y, x - \alpha t)e^{\gamma t} \tag{10}$$

with $f(\xi,\eta):\mathbb{R}^2\to\mathbb{R}$. Inserting (10) into (5) gives us

$$(U-\alpha)\frac{\partial}{\partial\eta}\Delta f+\gamma\Delta f-U''f_\eta=\nu\Delta\Delta f. \tag{11}$$

This equation admits the scaling symmetry $\tilde{X}_1 = f\partial_f$ as well as the translation symmetry $\tilde{X}_2=\partial_\eta$. We repeat the procedure leading to (10). We choose f to be invariant under the symmetry $\delta\tilde{X}_1+\tilde{X}_2$ and after renaming the constants, we get an ansatz which we can write as

$$\psi=f(y)e^{i\tilde{\alpha}(x-ct)} \tag{12}$$

with $\tilde{\alpha},c\in\mathbb{C}$. This ansatz leads directly to the Orr-Sommerfeld-Equation (13) [5, pp.424-429].

$$(U-c)\left(\frac{d^2}{dy^2}-\tilde{\alpha}^2\right)f=\frac{1}{i\tilde{\alpha}Re}\left(\frac{d^2}{dy^2}-\tilde{\alpha}^2\right)^2 f \tag{13}$$

In other words, the OSE is derived from a successive symmetry reduction of the linearized Navier Stokes Equation each time using the full set of admitted symmetries. Note that this holds true for both viscous and inviscid case and is usually referred to as normal mode or modal Ansatz.

2.2 Additional Symmetries for the Channel Flow

For an inviscid parabolic base flow of the form $U(y)=a\cdot(y+b)^2+d$, we found one additional symmetry:

$$X_4=(2d\cdot t-x)\partial_x-(y+b)\partial_y+t\partial_t \tag{14}$$

This symmetry does also occur in other formulations of problem (5), e.g. using velocities and pressure (3 equations) or using the stream function and pressure (2 equations).

3 The Channel Flow

3.1 Rescaling

In the present subsection we consider a parabolic channel flow of the form

$$U:[-h/2,h/2]\to\mathbb{R} \qquad y\to U_{max}\left(1-\left(\frac{2y}{h}\right)^2\right) \tag{15}$$

with $U_{max}=-\frac{\partial p}{\partial x}\frac{h^2}{8\eta}$ where η is the dynamic viscosity (see also [6, p.183]) We define a characteristic velocity and the respective Reynolds number as

$$u_\tau^2:=-\frac{\partial p}{\partial x}\cdot\frac{h}{\varrho} \quad , \quad Re_\tau:=\frac{u_\tau h}{\nu}, \tag{16}$$

where ν is the kinematic viscosity. We rescale the variables with

$$\tilde{x}_i := x_i \cdot \frac{2}{h} \quad , \quad \tilde{u}_i := u_i \cdot \frac{1}{U_{max}} \tag{17}$$

$$\tilde{p} := p \cdot \frac{1}{\varrho U_{max}^2} \quad , \quad \tilde{t} := t \cdot Re_\tau \frac{u_\tau}{4h} \quad , \quad \tilde{\psi} := \psi \cdot \frac{2}{U_{max} h}. \tag{18}$$

Then, equation (4) and (5) hold for the dimensionless variables. In place of ν, we then write $1/\mathrm{Re}$, where Re is the Reynolds number $\mathrm{Re} = \frac{U_{max} \cdot h}{2\nu}$. The base flow transforms to $U : [-1, 1] \to \mathbb{R}$ with

$$\tilde{U}(\tilde{y}) = 1 - \tilde{y}^2. \tag{19}$$

We see that the dimensionless time $\tilde{t}$ is obtained by multiplication of t with the Reynolds number Re_τ. Hence the gradient $\partial/\partial t$ scales with the Reynolds number. Thus for a possibly unstable flow the instability grows faster for high Reynolds numbers.

Henceforth, we will just work with the dimensionless formulation and all tildes will be omitted.

3.2 Invariant Solution

We consider equation (5) in the inviscid case ($\nu = 0$) with the base flow $U : [-1, 1] \to \mathbb{R} : y \to 1 - y^2$. This equation has the Lie-Point-Symmetries X_1, X_2, X_3 and X_4 (see (6)-(9) and (14)). We thus can formulate the following general symmetry

$$X := \alpha \partial_x + \delta \partial_t + \beta \psi \partial_\psi + \gamma((2t - x)\partial_x - y\partial_y + t\partial_t) \tag{20}$$

The case $\gamma = 0$ is not interesting for us as it is treated in the Orr-Sommerfeld-Equation. Hence we rescale X such that $\gamma = 1$. All invariant solutions ψ of (5) with $\nu = 0$ respective to (20) are of the following form:

$$\xi := (t + \delta)(x - t + \delta - \alpha) \tag{21}$$

$$\eta := y(\delta + t) \tag{22}$$

$$\psi := (\delta + t)^\beta f(\xi, \eta) \tag{23}$$

with $f : \mathbb{R}^2 \to \mathbb{R}$. Inserting ansatz (21)-(23) into equation (5) gives us the linear PDE of third order (24)

$$\left((\xi - \eta^2) \frac{\partial}{\partial \xi} + \eta \frac{\partial}{\partial \eta} + (2 + \beta) \right) \Delta f + 2 \frac{\partial f}{\partial \xi} = 0. \tag{24}$$

This equation has no further symmetries except the scaling symmetry in f. In the sequel, we analyze solutions which are invariant under (20) and present ways to find solutions which fulfill the boundary conditions of the channel flow.

3.3 Interpretation of Invariant Solution

Suppose we have a solution ψ which is invariant under symmetry (20). Then, the two velocities u and v can be written as

$$u = \frac{\partial \psi}{\partial y} = (\delta + t)^{\beta+1} \frac{\partial f}{\partial \eta} \tag{25}$$

$$v = -\frac{\partial \psi}{\partial x} = -(\delta + t)^{\beta+1} \frac{\partial f}{\partial \xi} \tag{26}$$

where ξ and η are defined in (21) and (22). We search for paths $(x(t), y(t))$ which satisfy $\xi(x(t), t) = \xi_0$ and $\eta(y(t), t) = \eta_0$, in other words which are constant on ξ and η. These paths are given by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \frac{\xi_0}{t+\delta} + t - \delta + \alpha \\ \frac{\eta_0}{t+\delta} \end{pmatrix}. \tag{27}$$

The linear component in $x(t)$ suggests a traveling wave solution. Applying the rescaling conditions shows that the travelling wave would have a velocity of U_{max}. The component $\frac{1}{t+\delta}$ suggests a solution with steepening gradients. On the paths given by (27), the amplitude of u, v increases with $(\delta + t)^{\beta+1}$ (see (25) and (26)). Thus, depending on β, on the paths (27) an algebraic growth or decay of the velocities can be observed.

With the steepening of gradients both in x- and y-direction the effect of viscosity increases in time. This is also inherently given in the equation which we get if we insert ansatz (21)-(23) into the dimensionless form of equation (5) with $\nu \neq 0$:

$$((\xi - \eta^2)\frac{\partial}{\partial \xi} + \eta \frac{\partial}{\partial \eta} + (2 + \beta))\Delta f + 2\frac{\partial f}{\partial \xi} = \frac{(\delta + t)^3}{Re} \Delta\Delta f. \tag{28}$$

Thus it seems useful to divide the life of the flow into a period where similarity and thus ansatz (23) holds, i.e. $(\delta + t)^3 \ll Re$, and into a later period where the gradients are too high for the inviscid assumption. Then, the viscous term on the right side of equation (28) equilibrates one or several terms on the left hand side. This is of course not fully valid since t appears as a parameter and similarity is broken in a strict sense. Still, the importance of viscosity becomes apparent.

Further inserting ansatz (21)-(23) into the fully non-linear equation (4) shows that the nonlinear terms decay algebraically with $\beta = -3$. If viscosity is also employed the limits for the temporal validity of the solution hold true as given above.

$$\begin{aligned} &\left((\xi - \eta^2)\frac{\partial}{\partial \xi} + \eta \frac{\partial}{\partial \eta} + (2 + \beta) \right) \Delta f + 2\frac{\partial f}{\partial \xi} \\ &\qquad - (\delta + t)^{\beta+3} \cdot \nabla \times (\nabla f \cdot \Delta f) = \frac{(\delta + t)^3}{Re} \Delta\Delta f \end{aligned} \tag{29}$$

It remains to be shown if it is possible to formulate an eigenvalue problem for β for the linear inviscid case. As a result, one could make a statement about the order of the algebraic growth or decay of disturbances.

3.4 Taylor Expansion

In the following section, we will present a possible procedure to solve (24) via Taylor expansion of f in η.

$$f(\xi,\eta) = \sum_{n=0}^{\infty} \eta^n a_n(\xi) \tag{30}$$

We insert this expansion into equation (24). Collecting powers in η gives an infinite system of ordinary differential equations of third order. For $n \in \{0,1\}$ we get

$$\begin{aligned} \xi a_n''' + (2+n+\beta)a_n'' - (n+1)(n-2)a_n' \\ +(n+1)(n+2)\xi a_{n+2}' + (n+1)(n+2)(2+n+\beta)a_{n+2} = &\quad 0 \end{aligned} \tag{31}$$

and for $n \geq 2$ we get

$$\begin{aligned} -a_{n-2}''' + \xi a_n''' + (2+n+\beta)a_n'' - (n+1)(n-2)a_n' \\ +(n+1)(n+2)\xi a_{n+2}' + (n+1)(n+2)(2+n+\beta)a_{n+2} = &\quad 0 \end{aligned} \tag{32}$$

The system of ODEs (31) and (32) can be splitted into one system of odd and one system of even indices. The systems of ODEs can iteratively be solved if $a_0(\xi)$ and $a_1(\xi)$ are known.

For even indices, we will give a semi-closed solution. Assume that $a_0(\xi) = \xi^m$ and $m + \beta + 1 \geq 1$. Then we get

$$a_2(\xi) := \frac{1}{2!}\left(\frac{m}{m+1+\beta}\cdot\xi^{m-1} - \frac{m(m-1)}{1}\cdot\xi^{m-2}\right) \tag{33}$$

and for $n \geq 2$

$$a_{2n}^m(\xi) := \sum_{k=1}^{n}\left(\left(\prod_{l=0}^{n+k-1}(m-l)\right)\left(\prod_{l=1}^{n-k}\frac{1}{m+\beta+l}\right) b_{n,k}\xi^{m-n-k}\right) \tag{34}$$

with the following iterative scheme for the coefficients $b_{n,k}$

$$b_{n+1,1} := (2n+1)(2n-2)b_{n,1} \tag{35}$$

$$b_{n+1,k} := (2n)(2n-1)b_{n-1,k-1} - b_{n,k-1} + (2n+1)(2n-2)b_{n,k} \tag{36}$$

$$b_{n+1,n+1} := -b_{n,n} \tag{37}$$

The first elements of $b_{n,k}$ are given by

$$b_{2,1} := 4, \qquad b_{2,2} := 1 \tag{38}$$

$$b_{3,1} := 16, \qquad b_{3,2} := -6, \qquad b_{3,4} := -1 \tag{39}$$

It remains to verify under which conditions for β, n and m the resulting series (30) converges uniformly. Furthermore, consistency with the initial conditions has to be investigated.

3.5 Boundary Conditions

For the channel flow with inviscid perturbations we have the boundary conditions $v|_{y=\pm 1} = 0$. Applying these conditions to ansatz (21)-(23) gives us $\frac{\partial f}{\partial \xi} = 0$ for all (ξ, η) (see also (26)). As a consequence, (24) degenerates to

$$\eta \frac{d^3}{d\eta^3} f + (2+\beta) \frac{d^2}{d\eta^2} f = 0. \tag{40}$$

For $\beta \notin \{-2, -1, 0\}$ (40) has the general solution $f = C_1 \eta^{-\beta} + C_2 \eta + C_3$. The resulting field of perturbations

$$u = -C_1 \beta y^{-\beta-1} + C_2 (\delta + t)^{\beta+1} \tag{41}$$

$$v = 0 \tag{42}$$

does not give any valuable results, as the first term of u is stationary and the second term does not depend on space. Furthermore, all velocities in y-direction vanish. As a consequence, we do not apply boundary conditions directly on ansatz (23), but on the superposition of solutions of (23) for different β.

As (5) is a linear equation, we can apply superposition. We say that our general solution ψ is the sum of invariant solutions ψ_i under the symmetries X_i with the 3-tuples $(\alpha, \beta, \delta)_i$

$$\psi = \sum_i \psi_i. \tag{43}$$

We use the Taylor expansion (30) for f_i

$$\psi_i = (\delta + t)^{\beta_i} \cdot f_i(\xi, \eta) \tag{44}$$

$$= (\delta + t)^{\beta_i} \cdot \sum_{n=0}^{\infty} \eta^n a_{i,n}(\xi). \tag{45}$$

Now, we can formulate conditions for $a_{i,n}(\xi_i)$. Suppose the series (43) converges uniformly. Then the boundary conditions can be rewritten as follows:

$$-v|_{y=1} = \sum_{n=0}^{\infty} \sum_i (\delta_i + t)^{\beta_i+n+1} a'_{in}(\xi_i) = 0 \tag{46}$$

$$-v|_{y=-1} = \sum_{n=0}^{\infty} (-1)^n \sum_i (\delta_i + t)^{\beta_i+n+1} a'_{in}(\xi_i) = 0 \tag{47}$$

Substraction and addition of (46) and (47) allows us to split boundary conditions into one condition for $a_{i,n}$ with even n and one condition for $a_{i,n}$ with odd n:

$$\sum_i \sum_{n=0}^{\infty} (\delta_i + t)^{\beta_i+1+2n} a'_{i,2n}(\xi_i) = 0 \tag{48}$$

$$\sum_i \sum_{n=0}^{\infty} (\delta_i + t)^{\beta_i+2+2n} a'_{i,2n+1}(\xi_i) = 0 \tag{49}$$

We define an equivalence relation $\sim$ by $\beta_i \sim \beta_j \Leftrightarrow (\beta_i - \beta_j) \in 2\mathbb{Z}$. The boundary conditions (48) and (49) act on each equivalence class separately. Assume that $\alpha_i = \alpha =$ const. and $\delta_i = \delta =$ const. Collecting powers in (48) and (49) gives us the following boundary conditions for every equivalence class $[\beta_0]$.

$$(\forall m \in \mathbb{Z}) \qquad \sum_{\substack{\beta_i - \beta_0 + 2n = m \\ 2n \geq 0 \\ \beta_i \sim \beta_0}} a'_{i,2n}(\xi) = 0 \tag{50}$$

$$(\forall m \in \mathbb{Z}) \qquad \sum_{\substack{\beta_i - \beta_0 + 2n = m \\ 2n \geq 0 \\ \beta_i \sim \beta_0}} a'_{i,2n+1}(\xi) = 0 \tag{51}$$

Thus, solving the superposition problem with boundary conditions reduces to finding $a_{i,0}(\xi)$ such that (50) and (51) are satisfied with and such that the Taylor expansion converges uniformly with the solution (34).

4 Conclusion

We analyzed the symmetries of the linearized equation for stability theory of parallel flows. We showed that there are three basic symmetries which lead to the Orr-Sommerfeld-Equation. We also showed that for the parabolic channel flow, there is one additional symmetry. This symmetry in combination with the three basic symmetries leads to a completely new Ansatz function. The algebraic growth of the perturbations as well as growth of the viscous terms is inherently given in this Ansatz. We gave one expansion of the invariant function and reformulated the boundary conditions with respect to this expansion.

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