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Multidimensional partial differential equation systems: Generating new systems via conservation laws, potentials, gauges, subsystems

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For many systems of partial differential equations (PDEs), including nonlinear ones, one can construct nonlocally related PDE systems. In recent years, such nonlocally related systems have proven to be useful in applications. In particular, they have yielded systematically nonlocal symmetries, nonlocal conservation laws, noninvertible linearizations, and new exact solutions for many different PDE systems of interest. However, the overwhelming majority of new results and theoretical understanding pertain only to PDE systems with two independent variables. The situation for PDE systems with more than two independent variables turns out to be much more complicated due to gauge freedom relating potential variables. The current paper, together with the companion paper [A. F. Cheviakov and G. W. Bluman, J. Math. Phys. 51, 103522 (2010)], synthesizes and systematically extends known results for nonlocally related systems arising for multidimensional PDE systems, i.e., for PDE systems with three or more independent variables. The presented framework includes potential systems arising from lower-degree conservation laws of a given PDE system. Nonlocally related multidimensional PDE systems are discussed in terms of their construction, properties, and applications.
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I. INTRODUCTION

For many disciplines, mathematical models involve nonlinear partial differential equations (PDEs), which usually cannot be solved analytically. For a given nonlinear PDE system, it is sometimes possible to use a transformation of variables that recasts the system into one that successfully yields to a particular method of analysis. In particular, this is often the case when the given and transformed PDE systems are *nonlocally related*, i.e., some variables in one of these systems depend nonlocally on the variables of the other system (for example, through their integrals).

A common example of a nonlocal variable is a potential variable. Potential variables are widely used in the analysis of models involving partial and ordinary differential equations, in relation to many applications, including mechanics, field theory, electromagnetism, and fluid dynamics. Such potential variables usually have a direct physical or geometrical meaning (e.g., in continuum mechanics, the relationship between the Eulerian and Lagrangian descriptions).

For any given system of PDEs with two independent variables, one can systematically construct nonlocally related potential systems and subsystems[2,3,14] having the same solution set as the given system. Due to nonlocal relations between solution sets, analysis of such nonlocally related

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systems can yield new results for the given system. In particular, one can systematically calculate nonlocal symmetries and nonlocal conservation laws, extend the construction of invariant and nonclassical solutions, obtain noninvertible linearizations, etc. This has led to new results for nonlinear wave and diffusion equations, equations of gas dynamics, continuum mechanics, electromagnetism, plasma equilibrium, etc.[2–13] [see also Ref. 14 (Chaps. 3–5) and references therein].

For PDE systems with two independent variables, the construction, properties, and use of nonlocally related PDE systems are relatively well-understood (see Sec. II).

This paper and the companion paper[1] are concerned with the systematic construction and use of nonlocally related PDE systems in multidimensions. For systems with $n>2$ independent variables, the situation for obtaining and using nonlocally related PDE systems is considerably more complex than in the two-dimensional case. In particular, every divergence-type conservation law gives rise to a vector potential subject to *gauge freedom*, i.e., which is defined to within arbitrary functions of the independent variables. The corresponding potential system is thus *underdetermined*. Additional equations involving potential variables, called *gauge constraints*, are needed to make such potential systems determined. In Ref. 11, it has been shown that only determined nonlocally related systems can yield nonlocal symmetries of a given PDE system.

Another important difference between two-dimensional and multidimensional PDE systems is that in higher dimensions, there exist several different types of conservation laws (divergence-type and *lower-degree* conservation laws). For example, in the case of $n=3$ independent variables, one can have a vanishing divergence or a vanishing curl; for $n>3$, $n-1$ types of conservation laws exist. (Lower-degree conservation laws arise naturally in the theory of the *variational bicomplex* applied to differential equations. For details, see Refs. 15–17.)

Due to the above-mentioned complexity, and additionally, due to the difficulty of performing computations for PDE systems involving many dependent and independent variables, few results have been obtained from using nonlocally related systems for multidimensional systems. To-date, obtained results include nonlocal symmetries for linear time-dependent wave and Maxwell's equations (in two and three spatial dimensions) with gauge constraints[11,12] and nonlocal symmetries for nonlinear magnetohydrodynamic (MHD) equilibrium equations in three spatial dimensions, following from a curl-type conservation law.[18,19] Another well-known example is the nonlocal Geroch symmetry group[20] for the Einstein equations possessing a Killing symmetry.

This paper attempts to synthesize and extend known results on nonlocal analysis of multidimensional PDE systems and, in particular, to generalize existing results for PDE systems with two independent variables to the multidimensional case.

The rest of this paper is organized as follows.

In Sec. II, we briefly review the nonlocal framework for PDE systems with two independent variables. In particular, for any PDE system with two independent variables, we review the construction of nonlocally related potential systems and subsystems, the construction of trees of nonlocally related PDE systems, and systematic procedures for seeking nonlocal conservation laws and nonlocal symmetries.

Sections III–VII are concerned with the nonlocal framework for PDE systems with three or more independent variables (although most formulas still hold for the case of two independent variables). In multidimensions, nonlocally related PDE systems can arise in three ways: (a) as potential systems from usual (divergence-type) conservation laws; (b) as potential systems from lower-degree conservation laws; and (c) as nonlocally related subsystems.

In Sec. III, divergence-type conservation laws for multidimensional PDE systems are studied. Through use of the direct construction method, one obtains conservation laws and the associated underdetermined potential systems. Furthermore, it is shown how to obtain determined potential systems through gauge constraints.

In Sec. IV, the notion of nonlocally related subsystems for PDE systems with two independent variables is generalized to multidimensional PDE systems.

Section V contains a general consideration of lower-degree conservation laws, including their forms, properties, and potential systems that arise from such conservation laws.

In Sec. VI, there is a discussion of the use of nonlocally related PDE systems to obtain

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nonlocal symmetries and nonlocal conservation laws of a given multidimensional PDE system. Important theorems for PDE systems in two dimensions are generalized to the multidimensional case.

Finally, Sec. VII contains a systematic procedure for the construction of sets (*trees*) of nonlocally related PDE systems in multidimensions.

Examples illustrating the framework presented in this paper are exhibited in the companion paper.[1]

II. NONLOCALLY RELATED PDE SYSTEMS IN TWO DIMENSIONS

A. Local conservation laws

Consider a PDE system $\mathbf{R}\{x,t;u\}$ of order k, with m dependent variables $u=(u^1,\ldots,u^m)$ and two independent variables $(x^1,x^2)=(x,t)$,

$$R^\sigma[u]=R^\sigma(x,t,u,\partial u,\ldots,\partial^k u)=0, \quad \sigma=1,\ldots,N. \tag{2.1}$$

Here ∂u denotes first order partial derivatives; $\partial^p u$ denotes pth order partial derivatives appearing in (2.1), $2\le p\le k$. In addition, we denote partial derivatives by $u_i^\mu=\partial u^\mu/\partial x^i$, $i=1,2$, and assume summation for repeated indices.

Local conservation laws of (2.1) are given by scalar divergence expressions

$$\mathrm{D}_t\Psi[u]+\mathrm{D}_x\Phi[u]=0 \tag{2.2}$$

for some density $\Psi[u]=\Psi(x,t,u,\partial u,\ldots,\partial^r u)$ and flux $\Phi[u]=\Phi(x,t,u,\partial u,\ldots,\partial^r u)$. In Eq. (2.2), the total derivative operators are given by

$$\mathrm{D}_t=\frac{\partial}{\partial t}+u_t\frac{\partial}{\partial u}+u_{xt}\frac{\partial}{\partial u_x}+u_{tt}\frac{\partial}{\partial u_t}+\ \ldots, \quad \mathrm{D}_x=\frac{\partial}{\partial x}+u_x\frac{\partial}{\partial u}+u_{xx}\frac{\partial}{\partial u_x}+u_{xt}\frac{\partial}{\partial u_t}+\ \ldots.$$

Definition 2.1: Local conservation law (2.2) of PDE system (2.1) is *trivial* if its fluxes are of the form $\Phi^i[u]=M^i[u]+H^i[u]$, where $M^i[u]$ and $H^i[u]$ are functions of x,u and derivatives of u, with $M^i[u]$ vanishing on the solutions of PDE system (2.1) and $\mathrm{D}_iH^i[u]\equiv 0$ divergence-free.

In particular, a trivial conservation law contains no information about given PDE system (2.1) and arises in two cases.

(1) Each of its fluxes vanishes identically on the solutions of given PDE system (2.1).
(2) The conservation law vanishes identically as a differential identity. In particular, this second type of trivial conservation law is simply an identity holding for arbitrary fluxes.

The notion of a trivial conservation law leads to the following definitions of equivalence and linear dependence of conservation laws.

Definition 2.2: Two conservation laws $\mathrm{D}_i\Phi^i[u]=0$ and $\mathrm{D}_i\Psi^i[u]=0$ are *equivalent* if $\mathrm{D}_i(\Phi^i[u]-\Psi^i[u])=0$ is a trivial conservation law. An *equivalence class* of conservation laws consists of all conservation laws equivalent to some given nontrivial conservation law.

Definition 2.3: A set of l conservation laws $\{\mathrm{D}_i\Phi^i_{(j)}[u]=0\}_{j=1}^l$ is *linearly dependent* if there exists a set of constants $\{a^{(j)}\}_{j=1}^l$, not all zero, such that the linear combination,

$$\mathrm{D}_i(a^{(j)}\Phi^i_{(j)}[u])=0,$$

is a trivial conservation law. In this case, up to equivalence, one of the conservation laws in the set can be expressed as a linear combination of the others.

Construction of local conservation laws. For any given PDE system (2.1), local conservation laws (2.2) can be systematically sought using the direct method.[14,21,22] Within this method, one takes a linear combination of the equations of system (2.1) with multipliers $\{\Lambda_\sigma[U]\}_{\sigma=1}^N=\{\Lambda_\sigma(x,t,U,\partial U,\ldots,\partial^l U)\}_{\sigma=1}^N$, depending on some prescribed independent and dependent variables and their derivatives to some finite order l, which yields a divergence expression

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$$\Lambda_\sigma[U]R^\sigma[U] \equiv \mathrm{D}_i\Phi^i[U] \tag{2.3}$$

holding for an *arbitrary* function $U(x,t)$. Then on the solutions $U(x,t)=u(x,t)$ of PDE system (2.1), one has the local conservation law

$$\Lambda_\sigma[u]R^\sigma[u] = \mathrm{D}_i\Phi^i[u] = 0. \tag{2.4}$$

Multipliers are found from the fact that the left-hand side of expression (2.3) depending on functions $U(x,t)=(U^1(x,t),\ldots,U^m(x,t))$ yields a divergence expression if and only if it is annihilated by the Euler operators,

$$\mathrm{E}_{U^j} = \frac{\partial}{\partial U^j} - \mathrm{D}_i\frac{\partial}{\partial U^j_i} + \cdots + (-1)^s\mathrm{D}_{i_1}\ldots\mathrm{D}_{i_s}\frac{\partial}{\partial U^j_{i_1\ldots i_s}} + \cdots, \quad i,i_\ell = 1,2, \tag{2.5}$$

with respect to each $U^j(x,t)$, $j=1,\ldots,m$. [For the case of $n\geq 3$ dimensions, in (2.5), $i,i_\ell=1,\ldots,n$.]

For PDE systems that can be written in a solved form with respect to some leading derivatives (which is the case for most physical systems), the direct method is *complete*, i.e., it yields all local conservation laws. The following theorem holds (Ref. 14, Chap. 1).

Theorem 2.1: *Suppose each PDE of given system (2.1) is written in a solved form*

$$R^\sigma[u] = u^{j_\sigma}_{i_{\sigma,1}\ldots i_{\sigma,s}} - G^\sigma(x,u,\partial u,\ldots,\partial^k u) = 0, \quad \sigma = 1,\ldots,N, \tag{2.6}$$

where $s\leq k$, $1\leq j_\sigma\leq m$, $1\leq i_{\sigma,1},\ldots,i_{\sigma,s}\leq 2$ for all $\sigma=1,\ldots,N$. In particular, in (2.6), $\{u^{j_\sigma}_{i_{\sigma,1}\ldots i_{\sigma,s}}\}$ is a set of N linearly independent sth order leading partial derivatives with the property that none of them or their differential consequences appears in $\{G^\sigma[u]\}_{\sigma=1}^N$. Then each local conservation law (2.2) of PDE system (2.1), up to conservation law equivalence, arises from the characteristic form

$$\mathrm{D}_i\Phi^i[U] = \Lambda_\sigma[U](U^{j_\sigma}_{i_{\sigma,1}\ldots i_{\sigma,s}} - G^\sigma[U]) = 0, \tag{2.7}$$

in terms of a set of local multipliers $\{\Lambda_\sigma[U]\}_{\sigma=1}^N$.

Remark 2.1: It is of importance to note that the conservation law structure of a given PDE system does not change under a point transformation of its dependent and independent variables. In particular, under such a transformation, local conservation laws are mapped into local conservation laws of the system.

B. Potential systems

Each conservation law (2.2) yields a pair of *potential equations*,

$$v_x = \Psi[u], \quad v_t = -\Phi[u] \tag{2.8}$$

for some auxiliary *potential variable* (potential) $v=v(x,t)$. A *potential system* $\mathbf{S}\{x,t;u,v\}$ is given by the union of given system (2.1) and potential Eqs. (2.8),

$$R^\sigma[u] = R^\sigma(x,t,u,\partial u,\ldots,\partial^k u) = 0, \quad \sigma = 1,\ldots,N,$$

$$v_x = \Psi[u],$$

$$v_t = -\Phi[u]. \tag{2.9}$$

[Redundant equations can be excluded from (2.9) or kept as needed.]

Potential system $\mathbf{S}\{x,t;u,v\}$ (2.9) has essentially the same solution set as that of given PDE system $\mathbf{R}\{x,t;u\}$ (2.1). In particular, if $u=\Theta(x,t)$ is a solution of (2.1), then due to the satisfaction of the integrability condition $v_{xt}=v_{tx}$, it follows that there is a corresponding solution $v=\Psi(x,t)$ of potential system (2.9), unique to within an arbitrary constant, i.e., if $(u,v)=(\Theta(x,t),\Psi(x,t))$ is a

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solution of potential system (2.9), then so is $(u,v)=(\Theta(x,t),\Psi(x,t)+C)$ for any constant C. Conversely, if $(u,v)=(\Theta(x,t),\Psi(x,t))$ solves potential system (2.9), then by projection, $u=\Theta(x,t)$ solves given PDE system (2.1). Consequently, through this relationship between their solution sets, potential system $\mathbf{S}\{x,t;u,v\}$ (2.9) is nonlocally equivalent to given PDE system $\mathbf{R}\{x,t;u\}$ (2.1), and the mapping that relates systems (2.9) and (2.1) is noninvertible.

If given system $\mathbf{R}\{x,t;u\}$ (2.1) has $q>1$ linearly independent conservation laws, one can naturally construct singlet, couplet, …, q-plet potential systems, involving, respectively, one, two, …, q potential variables. The following definition is helpful.

Definition 2.4: Suppose q linearly independent local conservation laws are known for a given PDE system $\mathbf{R}\{x,t;u\}$. In terms of the resulting potential variables $v^1,\ldots,v^q$, the set of all corresponding 2^q-1 potential systems is called a *combination potential system* $\mathbb{P}_{v^1\ldots v^q}$.

Remark 2.2: Normally, potential system $\mathbf{S}\{x,t;u,v\}$ (2.9) and given system $\mathbf{R}\{x,t;u\}$ (2.1) are nonlocally related, and the potential v is a nonlocal variable. However, in some special cases, it can happen that the potential variable defined by (2.8) is local, i.e., it is a function of the local variables $x,t,u,\partial u,\ldots$. For details and examples, see Ref. 23.

C. Nonlocally related subsystems

A second important way of obtaining PDE systems that are nonlocally related to a given PDE system $\mathbf{R}\{x,t;u\}$ is through the construction of appropriate subsystems. A subsystem of $\mathbf{R}\{x,t;u\}$ is a PDE system that can be obtained from $\mathbf{R}\{x,t;u\}$ by excluding one or more of its variables with the properties that (1) each solution of the subsystem yields a solution of $\mathbf{R}\{x,t;u\}$; and, conversely (2) each solution of $\mathbf{R}\{x,t;u\}$ yields a solution of the subsystem. Hence in this sense the subsystem is equivalent to $\mathbf{R}\{x,t;u\}$. Subsystems can naturally arise through the elimination of one or more of the given dependent variables of $\mathbf{R}\{x,t;u\}$ as well as through the elimination of one or more of the resulting dependent variables following a point transformation that involves an interchange of one or more of the dependent and independent variables of $\mathbf{R}\{x,t;u\}$.

As an example, let $\mathbf{R}\{x,t;u,v\}$ be the system of nonlinear telegraph equations given by

$$u_t-v_x=0,$$

$$v_t-F(u)u_x-G(u)=0, \tag{2.10}$$

where (u,v) are dependent variables, (x,t) are independent variables, and $F(u)$, $G(u)$ are arbitrary constitutive functions. For arbitrary $F(u)$ and $G(u)$, the dependent variable v may be excluded, using $v_{tx}=v_{xt}$, to obtain the subsystem $\underline{\mathbf{R}}\{x,t;u\}$ given by

$$u_{tt}-(F(u)u_x)_x-(G(u))_x=0. \tag{2.11}$$

Subsystem (2.11) is obviously nonlocally related to (2.10) since $\mathbf{R}\{x,t;u,v\}$ is a potential system of $\underline{\mathbf{R}}\{x,t;u\}$, with potential variable v.

D. Nonlocal conservation laws

Suppose a given PDE system $\mathbf{R}\{x,t;u\}$ has a potential system $\mathbf{S}\{x,t;u,v\}$ involving one or more potential variables $v=(v^1,\ldots,v^q)$.

It is easy to see that each local conservation law of $\mathbf{R}\{x,t;u\}$ (with fluxes and densities depending only on components of x, u and partial derivatives of u) is also a local conservation law of $\mathbf{S}\{x,t;u,v\}$. However, there may be local conservation laws of $\mathbf{S}\{x,t;u,v\}$,

$$\mathrm{D}_t\Psi[u,v]+\mathrm{D}_x\Phi[u,v]=0, \tag{2.12}$$

with fluxes and densities depending on components of x, u, v, and partial derivatives of u and v, that are *not* expressible as a linear combination of the local conservation laws of $\mathbf{R}\{x,t;u\}$. Such conservation laws are referred to as *nonlocal conservation laws* of the given PDE system $\mathbf{R}\{x,t;u\}$.

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It turns out that in order to find nonlocal conservation laws of $\mathbf{R}\{x,t;u\}$ arising as local conservation laws of its potential system $\mathbf{S}\{x,t;u,v\}$, it is necessary that the conservation law multipliers of the potential system have an essential dependence on its potential variable(s) v. The following theorem holds.[24]

Theorem 2.2: *Each conservation law of any potential system* $\mathbf{S}\{x,t;u,v\}$*, arising from multipliers that do not essentially depend on the potential variable(s) v, is equivalent to a local conservation law of given system* $\mathbf{R}\{x,t;u\}$ *(2.1).*

E. Trees of nonlocally related systems

Using the direct method for finding local conservation laws in conjunction with the use of potential systems and nonlocally related subsystems, one can systematically construct sets (*trees*) of PDE systems nonlocally related to any other system in a tree, including a given PDE system.[3,14] The procedure is briefly outlined as follows.

(1) Find a set of independent local conservation laws for the given PDE system $\mathbf{R}\{x,t;u\}$.
(2) Use the set of known local conservation laws to introduce n potential variables v^i. Construct the corresponding combination potential systems involving one or more potential variables.
(3) For the obtained potential systems, seek additional linearly independent conservation laws. (Normally seek multipliers that have an essential dependence on the potential variables. This must be the case if one has obtained all local conservation laws of the given PDE system.) Eliminate conservation laws that are linearly dependent on the set of previously known local conservation laws. Use the additional conservation laws to introduce further potential variables.
(4) Use the further potential variables to construct further potential systems.
(5) Repeat steps (3) and (4), until no further linearly independent conservation laws are found for any nonlocally related potential system.
(6) For all PDE systems obtained so far, generate nonlocally related subsystems as described above.

Such obtained PDE systems nonlocally related to a given PDE system have proven to be useful in many applications for obtaining new analytical results for the given system, such as nonlocal symmetries, nonlocal conservation laws, noninvertible linearizations, and new exact solutions. Many examples can be found in Ref. 14 and references therein.

F. Nonlocal symmetries

Suppose a system of PDEs $\mathbf{R}\{x,t;u\}$ has a potential system $\mathbf{S}\{x,t;u,v\}$, $v=(v^1,\ldots,v^q)$, that is invariant under the one-parameter (ϵ) Lie group of point transformations,

$$x^* = x + \epsilon\xi_S(x,t,u,v) + O(\epsilon^2), \quad t^* = t + \epsilon\tau_S(x,t,u,v) + O(\epsilon^2),$$

$$u^* = u + \epsilon\eta_S(x,t,u,v) + O(\epsilon^2), \quad v^* = v + \epsilon\zeta_S(x,t,u,v) + O(\epsilon^2), \tag{2.13}$$

with corresponding infinitesimal generator,

$$\mathrm{X} = \xi_S(x,t,u,v)\frac{\partial}{\partial x} + \tau_S(x,t,u,v)\frac{\partial}{\partial t} + \eta_S^\mu(x,t,u,v)\frac{\partial}{\partial u^\mu} + \zeta_S^p(x,t,u,v)\frac{\partial}{\partial v^p}. \tag{2.14}$$

If the infinitesimals $(\xi_S(x,t,u,v),\tau_S(x,t,u,v),\eta_S(x,t,u,v))$ have an essential dependence on v, then point symmetry (2.13) defines a nonlocal symmetry of $\mathbf{R}\{x,t;u\}$. In particular, infinitesimal generator (2.14) does not project onto an infinitesimal generator for a local symmetry of the given system $\mathbf{R}\{x,t;u\}$. Such a nonlocal symmetry is called a *potential symmetry* of $\mathbf{R}\{x,t;u\}$.

As discussed in Sec. I, potential symmetries have been found and successfully used in many nonlinear and linear PDE systems of physical interest.

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Nonlocal symmetries have also been shown to arise as local symmetries of nonlocally related subsystems. For example, this is the case for planar gas dynamics equations[3] and the nonlinear wave equation.[10]

III. DIVERGENCE-TYPE CONSERVATION LAWS IN MULTIDIMENSIONS AND RESULTING POTENTIAL SYSTEMS

In two dimensions, all conservation laws are of divergence-type. These are the most common conservation laws encountered in three or more dimensions.

As will be discussed at length in Sec. V, other types of conservation laws (lower-degree, curl-type) can arise for PDE systems with $n>2$ independent variables. In this section, we consider divergence-type conservation laws separately because of their commonality and particular importance for analysis and applications, such as the construction of advanced numerical algorithms (see, e.g., discussion in Ref. 25).

A. Divergence-type conservation laws and corresponding potential systems

Consider a system $\mathbf{R}\{x;u\}$ of N partial differential equations of order k with n independent variables $x=(x^1,\ldots,x^n)$ and m dependent variables $u(x)=(u^1(x),\ldots,u^m(x))$, given by

$$R^\sigma[u]=R^\sigma(x,u,\partial u,\ldots,\partial^k u)=0,\quad \sigma=1,\ldots,N. \tag{3.1}$$

From now on, we restrict our attention to the multidimensional case of $n\geq 3$ independent variables.

Definition 3.1: A divergence-type conservation law of PDE system (3.1) is a divergence expression of the form

$$\operatorname{div}\Phi[u]\equiv \mathrm{D}_i\Phi^i(x,u,\partial u,\ldots,\partial^r u)=0, \tag{3.2}$$

in terms of total derivative operators,

$$\mathrm{D}_i=\frac{\partial}{\partial x^i}+u^\mu_i\frac{\partial}{\partial u^\mu}+u^\mu_{ii_1}\frac{\partial}{\partial u^\mu_{i_1}}+u^\mu_{ii_1i_2}\frac{\partial}{\partial u^\mu_{i_1i_2}}+\cdots, \tag{3.3}$$

holding on solutions of a given PDE system $\mathbf{R}\{x;u\}$.

The notions of trivial, equivalent, and linearly dependent divergence-type conservation laws carry over from Definitions 2.1, 2.2, and 2.3.

Similar to the two-dimensional case, divergence-type conservation laws can be constructed systematically for any PDE system $\mathbf{R}\{x;u\}$, e.g., using the direct method (Sec. II A). Moreover, the completeness Theorem 2.1 directly generalizes to the multidimensional case, as follows.

Theorem 3.1: *Suppose each PDE of given PDE system (3.1) is written in a solved form,*

$$R^\sigma[u]=u^{j_\sigma}_{i_{\sigma,1}\cdots i_{\sigma,s}}-G^\sigma(x,u,\partial u,\ldots,\partial^k u)=0,\quad \sigma=1,\ldots,N, \tag{3.4}$$

where $s\leq k$, $1\leq j_\sigma\leq m$, $1\leq i_{\sigma,1},\ldots,i_{\sigma,s}\leq n$, $\sigma=1,\ldots,N$. In particular, in (3.4), $\{u^{j_\sigma}_{i_{\sigma,1}\cdots i_{\sigma,s}}\}$ is a set of N linearly independent sth order leading partial derivatives with the property that none of them or their differential consequences appears in $\{G^\sigma[u]\}_{\sigma=1}^N$. Then each local conservation law (3.2) of PDE system (3.1), up to conservation law equivalence, arises from the characteristic form

$$\mathrm{D}_i\Phi^i[U]=\Lambda_\sigma[U](U^{j_\sigma}_{i_{\sigma,1}\cdots i_{\sigma,s}}-G^\sigma[U]), \tag{3.5}$$

in terms of a set of local multipliers $\{\Lambda_\sigma[U]\}_{\sigma=1}^N$.

Conservation law (3.2) directly yields potential equations. For example, consider a PDE system $\mathbf{R}\{x,y,z;u\}$ in $\mathbb{R}^3$, $x=(x,y,z)$, which has a divergence-type conservation law,

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$$\text{div } \mathbf{\Phi}[u] = \Phi^1_x[u] + \Phi^2_y[u] + \Phi^3_z[u] = 0. \tag{3.6}$$

From (3.6) it immediately follows that $\mathbf{\Phi}[u]=\text{curl } \Gamma[u]$, where $\mathbf{\Gamma}[u]=(\mathbf{\Gamma}^1[u],\Gamma^2[u],\Gamma^3[u])$ is a vector potential, involving three scalar potential variables. The potential equations are given by

$$\Gamma^3_y[u] - \Gamma^2_z[u] = \Phi^1[u],$$

$$\Gamma^1_z[u] - \Gamma^3_x[u] = \Phi^2[u],$$

$$\Gamma^2_x[u] - \Gamma^1_y[u] = \Phi^3[u]. \tag{3.7}$$

Unlike in the two-dimensional situation, the system of potential equations (3.7) is *underdetermined*. In particular, the system of potential equations (3.7) is invariant under the transformations

$$\mathbf{\Gamma}[u] \to \mathbf{\Gamma}[u] + \text{grad } \phi(x,y,z), \tag{3.8}$$

where $\phi(x,y,z)$ is an arbitrary smooth function of its arguments. Thus, the system of potential equations (3.7) has *gauge freedom*.

An additional equation (called a *gauge constraint*) relating the potential variables in (3.7) is required to augment potential equations (3.7) in order to eliminate its gauge freedom. Such a gauge constraint has the property that the all solutions $u(x)$ of the given system $\mathbf{R}\{x,y,z;u\}$ involving conservation law (3.6) satisfy gauge-constrained potential equations (3.7). A PDE system containing the equations of the given system $\mathbf{R}\{x,y,z;u\}$, potential equations (3.7), and the gauge constraint is an example of a *determined potential system*. For example, one can show that the following are gauge constraints:

- divergence (Coulomb) gauge: $\text{div } \mathbf{\Gamma} \equiv \Gamma^1_x + \Gamma^2_y + \Gamma^3_z = 0$,
- spatial gauge: $\Gamma^k=0$, $k=1$ or 2 or 3,
- Poincaré gauge: $x\Gamma^1+y\Gamma^2+z\Gamma^3=0$.

If one of the coordinates in a given PDE system is time t, special gauges are frequently used, such as

- Lorentz gauge (in (2+1) dimensions, (t,x,y)): $\Gamma^1_t - \Gamma^2_x - \Gamma^3_y = 0$,
- Cronstrom gauge (in (2+1) dimensions, (t,x,y)): $t\Gamma^1 - x\Gamma^2 - y\Gamma^3 = 0$.

Each of the above-listed gauge constraints eliminates gauge freedom, in the sense that the potential variables no longer depend on arbitrary functions of the independent variables. A choice of gauge constraint will depend on a particular application. In many cases the choice of an appropriate gauge constraint is an open problem.

The same situation applies for any given PDE system (3.1) with $n \geq 3$ independent variables with divergence-type conservation law (3.2). From Poincaré's lemma it follows that there exist $\frac{1}{2}n(n-1)$ potential functions $v^{jk}=-v^{kj}(x)$ $(j,k=1,\ldots,n)$, components of an $n \times n$ antisymmetric tensor, such that the system of n *potential equations*,

$$\Phi^i[u] = \mathrm{D}_j v^{ij}, \quad i = 1, \ldots, n, \tag{3.9}$$

is equivalent to divergence expression (3.2). PDE system (3.9) generalizes three-dimensional curl expression (3.7). Note that for $n > 3$, the number of potential variables is $\frac{1}{2}n(n-1) > n$. Hence here PDE system (3.9) is even more underdetermined than in the situation for three-dimensional PDE system (3.7). In particular, here the gauge freedom is exhibited by invariance under the transformations

$$v^{ij} \to v^{ij} + \mathrm{D}_k w^{ijk}, \tag{3.10}$$

where w^{ijk} are $\frac{1}{6}n(n-1)(n-2)$ arbitrary functions that are components of a totally antisymmetric tensor. [In particular, for $n=3$, there is only one such free function, which corresponds to gauge

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invariance condition (3.8) for curls.] In other words, the system of potential equations (3.9) has an infinite number of point symmetries (*gauge symmetries*),

$$\mathrm{X}_{\text{gauge}} = \mathrm{D}_k w^{ijk} \frac{\partial}{\partial v^{ij}}. \tag{3.11}$$

The corresponding *potential system* $\mathbf{S}\{x;u,v\}$ is given by the union of the equations of $\mathbf{R}\{x;u\}$ (3.1) and potential equations (3.9). [Some of the equations of $\mathbf{R}\{x;u\}$ may be differential consequences of potential equations (3.9), and thus excluded from $\mathbf{S}\{x;u,v\}$.]

As in the two-dimensional case, it follows that the solution sets of a given system $\mathbf{R}\{x;u\}$ and its potential system $\mathbf{S}\{x;u,v\}$ are equivalent. In general, the potential variables v are nonlocal variables relative to $\mathbf{R}\{x;u\}$, and the PDE system $\mathbf{S}\{x;u,v\}$ is nonlocally related to $\mathbf{R}\{x;u\}$.

As it stands, the potential system $\mathbf{S}\{x;u,v\}$ is *underdetermined* due to its gauge freedom (3.10). A *determined potential system* is a union of a potential system $\mathbf{S}\{x;u,v\}$ and a set of one or more gauge constraints that eliminates the gauge freedom.

Remark 3.1: If a given PDE system $\mathbf{R}\{x;u\}$ has $q \geq 1$ independent divergence-type conservation laws (3.2), one can also consider k-plet potential systems, $k=1,\ldots,q$, which taken together form a combination potential system (Definition 2.4). In order to obtain a determined potential system, each k-plet potential system needs to be appended with a sufficient number of gauge constraints that eliminate gauge freedom for all potential variables.

IV. NONLOCALLY RELATED SUBSYSTEMS IN MULTIDIMENSIONS

Another direct way of finding nonlocally related PDE systems in higher dimensions is through subsystems that are obtained after elimination of dependent variables by differential operations. Similar to the situation in the two-dimensional case, in order to obtain a nonlocally related subsystem $\mathbf{U}\{x;u\}$ of a given PDE system $\mathbf{UV}\{x;u,v\}$, it is obviously necessary that a dependent variable v only occurs in $\mathbf{UV}\{x;u,v\}$ in terms of its derivatives.

It should be noted that the construction of nonlocally related subsystems involves no gauge constraints since such subsystems are already determined. As will be shown later, this significantly simplifies nonlocal symmetry computations.

As a first example, consider the time-independent PDE system $\mathbf{VP}\{x,y,z;v^1,v^2,v^3,p\}$ of Euler equations of an inviscid, constant density fluid flow in three dimensions, which can be written as

$$\mathbf{v} \times (\text{curl } \mathbf{v}) = \text{grad}\left(\frac{p}{\rho} + \frac{1}{2}|\mathbf{v}|^2\right),$$

$$\text{div } \mathbf{v} = 0. \tag{4.1}$$

Here $\mathbf{v}=(v^1,v^2,v^3)$ is the fluid velocity vector and ρ=const the fluid density. In PDE system (4.1), one can exclude the pressure p by taking the curl of the vector equation. The resulting subsystem $\mathbf{V}\{x,y,z;v^1,v^2,v^3\}$ given by

$$\text{curl}[\mathbf{v} \times (\text{curl } \mathbf{v})] = 0,$$

$$\text{div } \mathbf{v} = 0 \tag{4.2}$$

is equivalent and nonlocally related to the Euler system $\mathbf{VP}\{x,y,z;v^1,v^2,v^3,p\}$.

As a second example, consider the PDE system $\mathbf{UV}\{x,y,t;u,v^1,v^2\}$ in one time and two space dimensions, given by

$$\mathbf{v}_t = \text{grad } u,$$

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$$u_t = K(|\mathbf{v}|)\text{div } \mathbf{v}. \tag{4.3}$$

In (4.3), $\mathbf{v}=(v^1,v^2)$ is a vector function, and $K(|\mathbf{v}|)$ is a constitutive function of the indicated scalar argument. PDE system (4.3) has the nonlocally related subsystem $\mathbf{V}\{x,y,t;v^1,v^2\}$, given by

$$\mathbf{v}_{tt} = \text{grad}[K(|\mathbf{v}|)\text{div } \mathbf{v}]. \tag{4.4}$$

In the companion paper,[1] subsystem (4.4) is used to obtain a nonlocal symmetry of PDE system (4.3) for a particular form of the constitutive function $K(|\mathbf{v}|)$.

V. LOWER-DEGREE CONSERVATION LAWS AND RELATED POTENTIAL SYSTEMS

In three or more dimensions, conservation laws are not limited to independent divergence expressions (3.2). For example, in three-dimensional space, a PDE system $\mathbf{R}\{x,y,z;u\}$ may have a vector conservation law given by

$$\text{curl } \mathbf{\Psi}[u] = 0, \tag{5.1}$$

where $\mathbf{\Psi}=(\Psi^1,\Psi^2,\Psi^3)$ is some flux vector depending on independent variables (x,y,z) and dependent variables u. Such a curl-type conservation law is often referred to as a *lower-degree conservation law*.[26]

Of course, conservation law (5.1) can be viewed as three divergence-type conservation laws corresponding to the three components of a curl. Accordingly, one can introduce a total of nine potential variables, with gauge constraints to be chosen. However, another (and in many ways more efficient) representation of a curl-free vector field is in terms of the gradient of a scalar function, i.e., conservation law (5.1) is equivalent to the set of potential equations,

$$\Psi^1[u] = w_x,$$

$$\Psi^2[u] = w_y,$$

$$\Psi^3[u] = w_z. \tag{5.2}$$

In (5.2), the nonlocal potential variable $w(x,y,z)$ is defined to within a constant. Hence the corresponding potential system is determined and requires no gauge constraints.

For PDE systems with three independent variables, the only possible types of conservation laws are of divergence-type and curl-type. Examples include PDE systems describing static electromagnetic fields, irrotational fluid dynamics, and ideal plasma equilibria. An example of the use of a nonlocally related potential system arising from a curl-type conservation law is presented in the companion paper.[1]

PDE systems with $n>3$ independent variables can have other types of *lower-degree conservation laws*.[26] Forms and properties of such conservation laws and the construction of corresponding potential systems are discussed later in this section. In particular, PDE systems with n independent variables can have $n-1$ types of conservation laws. Similar to conservation law (5.1), lower-degree conservation laws are expressed by several components, i.e., vanishing divergence expressions. It is important to note that lower-degree conservation laws can yield a smaller number of potential variables than divergence-type conservation laws, and thus require fewer gauge constraints. In particular, conservation laws of degree 1 (which generalize curl-type conservation laws in $n\geq 3$ dimensions) are shown to always yield determined potential equations, requiring no gauge constraints. Several examples of potential systems following from lower-degree conservation laws that arise in applications are presented in Ref. 1.

For the description of lower-degree conservation laws and arising potential systems, it is convenient to use differential form notation (e.g., Ref. 27). Note that formulas given in this section hold for $n\geq 2$ independent variables.

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TABLE I. Bases and dimensions of spaces of r-forms in $\mathbb{R}^n$.

r	Basis	Number of components
0	$\{1\}$	1
1	$\{\mathrm{d}x^{\mu_1}\}$	n
2	$\{\mathrm{d}x^{\mu_1}\wedge \mathrm{d}x^{\mu_2}\}$	$\frac{n(n-1)}{2}$
3	$\{\mathrm{d}x^{\mu_1}\wedge \mathrm{d}x^{\mu_2}\wedge \mathrm{d}x^{\mu_3}\}$	$\frac{n(n-1)(n-2)}{6}$
$\vdots$	$\vdots$	$\vdots$
$n-2$	$\{\mathrm{d}x^{\mu_1}\wedge \ldots \wedge \mathrm{d}x^{\mu_{n-2}}\}$	$\frac{n(n-1)}{2}$
$n-1$	$\{\mathrm{d}x^{\mu_1}\wedge \ldots \wedge \mathrm{d}x^{\mu_{n-1}}\}$	n
n	$\{\mathrm{d}x^{\mu_1}\wedge \ldots \wedge \mathrm{d}x^{\mu_n}\}$	1

A. Differential-geometric notation

Suppose $x=(x^1,\ldots,x^n)$ is a set of orthogonal coordinates in $\mathbb{R}^n$. Denote the corresponding basis by $\{\partial/\partial x^i\}_{i=1}^n$ and dual basis by $\{\mathrm{d}x^i\}_{i=1}^n$. Then any vector $\mathbf{v}\in\mathbb{R}^n$ and a covector κ have basis expansions

$$\mathbf{v}=v^i\mathbf{e}_i=v^i\frac{\partial}{\partial x^i},\qquad \kappa=\kappa_i dx^i. \tag{5.3}$$

A tensor of type (q,r) is a multilinear expression given by

$$T=T^{\mu_1,\ldots,\mu_q}_{\nu_1,\ldots,\nu_r}\frac{\partial}{\partial x^{\mu_1}}\otimes\ \ldots\ \otimes\frac{\partial}{\partial x^{\mu_q}}\otimes dx^{\nu_1}\otimes\ \ldots\ \otimes dx^{\nu_r}. \tag{5.4}$$

Consider a permutation $P(\mu_1,\ldots,\mu_r)=(P(\mu_1),\ldots,P(\mu_r))$ of integer indices $(\mu_1,\ldots,\mu_r)$. Let $\mathrm{sgn}(P)$ be the sign of the permutation P, and let S_r be the complete set of possible permutations of $(\mu_1,\ldots,\mu_r)$. The wedge product of r covector-space coordinates $\mathrm{d}x^i$ is given by

$$\mathrm{d}x^{\mu_1}\wedge\ \ldots\ \wedge\mathrm{d}x^{\mu_r}=\sum_{P\in S_r}\mathrm{sgn}(P)\mathrm{d}x^{\mu_{P(1)}}\otimes\ \ldots\ \otimes\mathrm{d}x^{\mu_{P(r)}}. \tag{5.5}$$

Wedge product (5.5) is totally antisymmetric, since the permutation of any two of its indices μ_i and μ_j leads to a change of sign.

A *differential form of order* r (an r-form) is given by

$$\omega(r)=\frac{1}{r!}\omega_{\mu_1\ldots\mu_r}\mathrm{d}x^{\mu_1}\wedge\ \ldots\ \wedge\mathrm{d}x^{\mu_r}, \tag{5.6}$$

where $\omega_{\mu_1\mu_2\ldots\mu_r}$ are components of a totally antisymmetric tensor of type $(0,r)$, and summation in all μ_i from 1 to n is assumed. The number of independent components of ω is given by the number of choices of r elements out of n, i.e., $\binom{n}{r}$. For example, in $\mathbb{R}^3$ with coordinates (x,y,z), 0- to 3-forms are given by

$$\omega^{(0)}=f,\qquad \omega^{(1)}=\omega_1\mathrm{d}x+\omega_2\mathrm{d}y+\omega_3\mathrm{d}z,$$

$$\omega^{(2)}=\omega_{12}\mathrm{d}x\wedge\mathrm{d}y+\omega_{23}\mathrm{d}y\wedge\mathrm{d}z+\omega_{31}\mathrm{d}z\wedge\mathrm{d}x,\qquad \omega^{(3)}=\omega_{123}\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z,$$

where $f,\omega_i,\omega_{ij},\omega_{ijk}$ are functions of x,y,z.

Dimensions and bases of spaces of r-forms in $\mathbb{R}^n$ are given in Table I, where each index μ_i

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independently takes on all values from 1 to n.

An *exterior derivative* of r-form (5.6) is an $r+1$-form given by

$$d\omega^{(r)} = \frac{1}{r!}\left(\frac{\partial}{\partial x^{\nu}}\omega_{\mu_1\ldots\mu_r}\right)dx^{\nu}\wedge dx^{\mu_1}\wedge \ldots \wedge dx^{\mu_r}. \tag{5.7}$$

In particular, in $\mathbb{R}^3$ with coordinates (x,y,z), the action of d is given by

$$d\omega^{(0)} = \frac{\partial f}{\partial x}dx + \frac{\partial f(x)}{\partial y}dy + \frac{\partial f(x)}{\partial z}dz,$$

$$d\omega^{(1)} = \left(\frac{\partial \omega_2}{\partial x} - \frac{\partial \omega_1}{\partial y}\right)dx\wedge dy + \left(\frac{\partial \omega_3}{\partial y} - \frac{\partial \omega_2}{\partial z}\right)dy\wedge dz + \left(\frac{\partial \omega_1}{\partial z} - \frac{\partial \omega_3}{\partial x}\right)dz\wedge dx,$$

$$d\omega^{(2)} = \left(\frac{\partial \omega_{23}}{\partial x} + \frac{\partial \omega_{31}}{\partial y} + \frac{\partial \omega_{12}}{\partial z}\right)dx\wedge dy\wedge dz,$$

$$d\omega^{(3)} = 0,$$

with the first three formulas corresponding, respectively, to the action of the vector calculus operations of "grad," "curl," and "div."

A differential form $\omega^{(r)}$ is *closed* if $d\omega^{(r)}=0$. A differential form $\omega^{(r)}$ is *exact* if $\omega^{(r)}=d\omega^{(r-1)}$ for some $\omega^{(r-1)}$. Because $d^2=0$, any exact form is automatically closed. From *Poincaré's lemma*, it also follows that every closed differential form $\omega^{(r)}$ in an open domain is locally exact (provided that the coefficients are sufficiently smooth).

B. Conservation laws of degree *r*. Lower-degree conservation laws

Consider PDE system $\mathbf{R}\{x;u\}$ (3.1) with $n\geq 2$ independent variables.

From now on, components of differential forms denoted by $\omega_{\mu_1\ldots\mu_r}[U]$ are assumed to depend on x, U, and derivatives of U. Hence, where necessary, differentiations by x^i are replaced by total derivative operators D_i.

Definition 5.1: A conservation law of degree r $(1\leq r\leq n-1)$ of PDE system $\mathbf{R}\{x;u\}$ (3.1) is given by an r-form $\omega^{(r)}[U]$ (5.6), such that its exterior derivative

$$\Omega^{(r+1)}[u] = d\omega^{(r)}[u] = 0 \tag{5.8}$$

on all solutions $U=u$ of a given PDE system $\mathbf{R}\{x;u\}$,[26]

$$\Omega_{\nu\mu_1\ldots\mu_r}[u]dx^{\nu}\wedge dx^{\mu_1}\wedge \ldots \wedge dx^{\mu_r} \equiv \left(\frac{\partial}{\partial x^{\nu}}\omega_{\mu_1\ldots\mu_r}[u]\right)dx^{\nu}\wedge dx^{\mu_1}\wedge \ldots \wedge dx^{\mu_r} = 0. \tag{5.9}$$

To clarify the structure of expression (5.9), let

$$B^i = \{\beta_1^i, \ldots, \beta_{r+1}^i\} \subset \{1,2,\ldots,n\}, \quad \beta_1^i < \ldots < \beta_{r+1}^i, \quad i = 1,\ldots,\binom{n}{r+1}.$$

Each B^i is an ordered subset of $r+1$ indices from the set $\{1,\ldots,n\}$. Let S^i be the set of permutations of B^i. Then in (5.9), a sum of terms with indices $(\nu,\mu_1,\ldots,\mu_r)$ that are a permutation of some B^i should vanish separately for each B^i,

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$$\sum_{\{\nu,\mu_1,\ldots,\mu_r\}\subset S^i} \mathrm{sgn}(\nu,\mu_1,\ldots,\mu_r)\frac{\partial}{\partial x^\nu}\omega_{\mu_1\cdots\mu_r}[u]=0, \quad i=1,\ldots,\binom{n}{r+1}, \tag{5.10}$$

where $\mathrm{sgn}(\nu,\mu_1,\ldots,\mu_r)$ is the sign of the permutation $(\nu,\mu_1,\ldots,\mu_r)$ of indices of B_i. Hence the following lemma holds.

Lemma 5.1: For a PDE system with n independent variables, each conservation law (5.9) of degree r is given by a set of $\binom{n}{r+1}$ *divergence expressions (5.10).*

Remark 5.1: A conservation law of degree r (5.9) has $\binom{n}{r}$ fluxes given by the components $\omega_{\mu_1\mu_2\cdots\mu_r}[U]$ of the differential form $\omega^{(r)}[U]$.

1. Conservation laws of degree r=n−1 in $\mathbb{R}^n$

Conservation laws of degree $r=n-1$ are simply classical divergence-type conservation laws (3.2). In particular, for $r=n-1$, differential form (5.6) can be written as

$$\omega^{(n-1)}[U]=\frac{1}{(n-1)!}\omega_{\mu_1\cdots\mu_{n-1}}[U]\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_{n-1}}=\varepsilon_{\nu\mu_1\cdots\mu_{n-1}}\Phi^\nu[U]\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_{n-1}}, \tag{5.11}$$

where $\varepsilon_{\nu\mu_1\mu_2\cdots\mu_{n-1}}$ is the n-dimensional Levi–Civita symbol [$\varepsilon_{\nu,\mu_1,\cdots\mu_{n-1}}=1$ if $(\nu,\mu_1\ldots\mu_{n-1})$ is an even permutation of $1,2,\ldots,n$, $\varepsilon_{\nu,\mu_1,\cdots\mu_{n-1}}=-1$ if $(\nu,\mu_1,\ldots,\mu_{n-1})$ is an odd permutation of $1,2,\ldots,n$, $\varepsilon_{\nu,\mu_1,\cdots\mu_{n-1}}=0$ if two or more indices coincide]. The vanishing of the exterior derivative on solutions $U=u$ of $\mathbf{R}\{x;u\}$ (3.1), i.e.,

$$d\omega^{(n-1)}[u]=(\mathrm{D}_\nu\Phi^\nu[u])dx^1\wedge\ldots\wedge dx^n=0,$$

yields the vanishing divergence $\mathrm{D}_\nu\Phi^\nu[u]=0$.

In particular, for the case $n=3$ and a PDE system $\mathbf{R}\{x,y,z;u\}$, the fluxes of a conservation law of degree two are given by the components of a differential 2-form,

$$\omega^{(2)}[U]=\Phi^1[U]\mathrm{d}y\wedge\mathrm{d}z+\Phi^2[U]\mathrm{d}z\wedge\mathrm{d}x+\Phi^3[U]\mathrm{d}x\wedge\mathrm{d}y. \tag{5.12}$$

The exterior derivative of (5.12) on solutions $U=u$ of the PDE system $\mathbf{R}\{x,y,z;u\}$ is given by

$$\mathrm{d}\omega^{(2)}[u]=(\mathrm{D}_x\Phi^1[u]+\mathrm{D}_y\Phi^2[u]+\mathrm{D}_z\Phi^3[u])\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z=0,$$

which yields a vanishing divergence $\mathrm{D}_x\Phi^1[u]+\mathrm{D}_y\Phi^2[u]+\mathrm{D}_z\Phi^3[u]=0$.

Similarly, for a PDE system with $n=4$ independent variables $(x^1,x^2,x^3,x^4)=(x,y,z,w)$, the fluxes of a degree three conservation law are the components of a 3-form,

$$\omega^{(3)}[U]=\Phi^1[U]\mathrm{d}y\wedge\mathrm{d}z\wedge\mathrm{d}w-\Phi^2[U]\mathrm{d}z\wedge\mathrm{d}w\wedge\mathrm{d}x+\Phi^3[U]\mathrm{d}w\wedge\mathrm{d}x\wedge\mathrm{d}y-\Phi^4[U]\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z. \tag{5.13}$$

The conservation law itself is given by

$$\mathrm{d}\omega^{(3)}[u]=\mathrm{D}_x\Phi^1[u]+\mathrm{D}_y\Phi^2[u]+\mathrm{D}_z\Phi^3[u]+\mathrm{D}_w\Phi^4[u]=0. \tag{5.14}$$

2. Lower-degree conservation laws

Definition 5.2: A *lower-degree conservation law* of a PDE system $\mathbf{R}\{x;u\}$ (3.1) is a conservation law (5.8) of degree $r<n-1$.

The notions of trivial, equivalent, and linearly dependent lower-degree conservation laws carry over from Definitions 2.1, 2.2, and 2.3.

As a first example, consider a PDE system $\mathbf{R}\{x,y,z;u\}$ with $n=3$ independent variables (x,y,z). The only lower-degree conservation law in $\mathbb{R}^3$ is the conservation law of degree 1. Its fluxes are the independent components of a differential 1-form,

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$$\omega^{(1)}[U] = \omega_1[U]\mathrm{d}x + \omega_2[U]\mathrm{d}y + \omega_3[U]\mathrm{d}z, \tag{5.15}$$

and the conservation law itself is given by the components of a differential 2-form,

$$\mathrm{d}\omega^{(1)}[u] = (\mathrm{D}_y\omega_3[u] - \mathrm{D}_z\omega_2[u])\mathrm{d}y \wedge \mathrm{d}z + (\mathrm{D}_z\omega_1[u] - \mathrm{D}_x\omega_3[u])\mathrm{d}z \wedge \mathrm{d}x + (\mathrm{D}_x\omega_2[u] - \mathrm{D}_y\omega_1[u])\mathrm{d}x \wedge \mathrm{d}y = 0,$$

i.e., three divergence expressions,

$$\mathrm{D}_y\omega_3[u] - \mathrm{D}_z\omega_2[u] = 0, \quad \mathrm{D}_z\omega_1[u] - \mathrm{D}_x\omega_3[u] = 0, \quad \mathrm{D}_x\omega_2[u] - \mathrm{D}_y\omega_1[u] = 0, \tag{5.16}$$

which are the components of the vector equation $\mathrm{curl}(\omega_1[u], \omega_2[u], \omega_3[u]) = 0$.

As a second example, consider the case of $n=4$ independent variables $(x^1, x^2, x^3, x^4) = (x, y, z, w)$. In addition to divergence-type (degree three) conservation laws, here one can have lower-degree conservation laws of orders 1 and/or 2. In particular, the fluxes of a conservation law of degree 1 are the independent components of a 1-form,

$$\omega^{(1)}[U] = \omega_1[U]\mathrm{d}x + \omega_2[U]\mathrm{d}y + \omega_3[U]\mathrm{d}z + \omega_4[U]\mathrm{d}w, \tag{5.17}$$

and the conservation law itself is given by the components of a differential 2-form,

$$\mathrm{d}\omega^{(1)}[u] = (\mathrm{D}_x\omega_2[u] - \mathrm{D}_y\omega_1[u])\mathrm{d}x \wedge \mathrm{d}y + (\mathrm{D}_x\omega_3[u] - \mathrm{D}_z\omega_1[u])\mathrm{d}x \wedge \mathrm{d}z + (\mathrm{D}_x\omega_4[u] - \mathrm{D}_w\omega_1[u])\mathrm{d}x \wedge \mathrm{d}w + (\mathrm{D}_y\omega_3[u] - \mathrm{D}_z\omega_2[u])\mathrm{d}y \wedge \mathrm{d}z + (\mathrm{D}_y\omega_4[u] - \mathrm{D}_w\omega_2[u])\mathrm{d}y \wedge \mathrm{d}w + (\mathrm{D}_z\omega_4[u] - \mathrm{D}_w\omega_3[u])\mathrm{d}z \wedge \mathrm{d}z = 0,$$

i.e., six divergence expressions,

$$\mathrm{D}_x\omega_2[u] - \mathrm{D}_y\omega_1[u] = 0, \quad \mathrm{D}_x\omega_3[u] - \mathrm{D}_z\omega_1[u] = 0,$$

$$\mathrm{D}_x\omega_4[u] - \mathrm{D}_w\omega_1[u] = 0, \quad \mathrm{D}_y\omega_3[u] - \mathrm{D}_z\omega_2[u] = 0,$$

$$\mathrm{D}_y\omega_4[u] - \mathrm{D}_w\omega_2[u] = 0, \quad \mathrm{D}_z\omega_4[u] - \mathrm{D}_w\omega_3[u] = 0. \tag{5.18}$$

For a conservation law of degree 2 in $\mathbb{R}^4$, fluxes are represented by the independent components of a 2-form,

$$\omega^{(2)}[U] = \omega_{12}[U]\mathrm{d}x \wedge \mathrm{d}y + \omega_{13}[U]\mathrm{d}x \wedge \mathrm{d}z + \omega_{14}[U]\mathrm{d}x \wedge \mathrm{d}w + \omega_{23}[U]\mathrm{d}y \wedge \mathrm{d}z + \omega_{24}[U]\mathrm{d}y \wedge \mathrm{d}w + \omega_{34}[U]\mathrm{d}z \wedge \mathrm{d}w. \tag{5.19}$$

The conservation law of degree 2 itself is given by the components of $\mathrm{d}\omega^{(2)}[u]=0$, i.e., the four divergence expressions,

$$\mathrm{D}_z\omega_{12}[u] - \mathrm{D}_y\omega_{13}[u] + \mathrm{D}_x\omega_{23}[u] = 0, \quad \mathrm{D}_w\omega_{12}[u] - \mathrm{D}_y\omega_{14}[u] + \mathrm{D}_x\omega_{24}[u] = 0,$$

$$\mathrm{D}_w\omega_{13}[u] - \mathrm{D}_z\omega_{14}[u] + \mathrm{D}_x\omega_{34}[u] = 0, \quad \mathrm{D}_w\omega_{23}[u] - \mathrm{D}_z\omega_{24}[u] + \mathrm{D}_y\omega_{34}[u] = 0. \tag{5.20}$$

The numbers of flux variables and scalar divergence expressions defining conservation laws of degree $r \le n-1$ in $\mathbb{R}^n$ are listed in Table II.

C. Potential equations

It is now shown how each conservation law of degree r, $1 \le r \le n-1$, directly yields a set of potential equations. From Poincaré's lemma, it follows that since $\mathrm{d}\omega^{(r)}[u]=0$ on solutions of $\mathbf{R}\{x;u\}$, one locally has $\omega^{(r)}[u] = \mathrm{d}\tilde{\omega}^{(r-1)}[u]$, for some $(r-1)$-form $\tilde{\omega}^{(r-1)}[u]$. Hence the potential equations are given by

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TABLE II. Numbers of conservation law components, potential equations, and potential variables, for conservation law (5.9) of degree r ($1 \leq r \leq n-1$).

CL degree r	# of CL components (divergence expressions)	# of potential equations (=# of CL fluxes)	# of potential variables
1	$\frac{n(n-1)}{2}$	n	1
2	$\frac{n(n-1)(n-2)}{6}$	$\frac{n(n-1)}{2}$	n
$\vdots$	$\vdots$	$\vdots$	$\vdots$
r	$\binom{n}{r+1}$	$\binom{n}{r}$	$\binom{n}{r-1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-2$	n	$\frac{n(n-1)}{2}$	$\frac{n(n-1)(n-2)}{6}$
$n-1$	1	n	$\frac{n(n-1)}{2}$

$$\frac{1}{r!}\omega_{\mu_1\ldots\mu_r}[u]\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_r}=\frac{1}{(r-1)!}\left(\frac{\partial}{\partial x^{\nu}}\tilde{\omega}_{\nu_1\ldots\nu_{r-1}}[u]\right)\mathrm{d}x^{\nu}\wedge\mathrm{d}x^{\nu_1}\wedge\ldots\wedge\mathrm{d}x^{\nu_{r-1}}. \tag{5.21}$$

Remark 5.2: One can show that potential equations (5.21) can be explicitly written as

$$\omega_{\mu_1\ldots\mu_r}[u]=\sum_{i=1}^{r}(-1)^{i-1}\frac{\partial}{\partial x^{\mu_i}}\tilde{\omega}_{\mu_1\ldots\overline{\mu_i}\ldots\mu_r}[u], \tag{5.22}$$

where $\mu_1<\ldots<\mu_r$ is one of the $\binom{n}{r}$ ordered r-element subsets of $\{1,\ldots,n\}$, and the subscript $\mu_1\ldots\overline{\mu_i}\ldots\mu_r$ denotes the ordered set of $r-1$ indices $\mu_1,\ldots,\mu_r$ when μ_i is not present. No summation is assumed in (5.22), except for the explicitly indicated sum in i.

The following definition is useful.

Definition 5.3: Suppose PDE system $\mathbf{R}\{x;u\}$ (3.1) has a conservation law of degree r. The corresponding *potential system of degree* r is a PDE system $\mathrm{S}\{x;u,\tilde{\omega}\}$ given by the union of the equations of $\mathbf{R}\{x;u\}$ and potential equations (5.22).

As per Table II, a potential system of degree r involves $\binom{n}{r}$ potential equations (5.21) for $\binom{n}{r-1}$ potential variables, which are independent components of the $(r-1)$-form $\tilde{\omega}[u]$.

As examples, we consider potential equations arising from all possible conservation laws in $\mathbb{R}^3$ and $\mathbb{R}^4$ and potential equations arising from "marginal," i.e., degrees 1 and $n-1$, conservation laws in $\mathbb{R}^n$.

1. Potential equations in $\mathbb{R}^3$

Consider a PDE system $\mathbf{R}\{x,y,z;u\}$ in $\mathbb{R}^3$. First, suppose $\mathbf{R}\{x,y,z;u\}$ has a conservation law of degree $r=2$, i.e., divergence-type conservation law (3.2) with corresponding differential form $\omega^{(2)}[U]$ given by (5.12). Using (5.22), one obtains three potential equations given by

$$\omega_{12}[u]\equiv\Phi^3[u]=\mathrm{D}_x\tilde{\omega}_2[u]-\mathrm{D}_y\tilde{\omega}_1[u],$$

$$\omega_{13}[u]\equiv-\Phi^2[u]=\mathrm{D}_x\tilde{\omega}_3[u]-\mathrm{D}_z\tilde{\omega}_1[u],$$

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$$\omega_{23}[u] \equiv \Phi^1[u] = \mathrm{D}_y\tilde{\omega}_3[u] - \mathrm{D}_z\tilde{\omega}_2[u]. \tag{5.23}$$

Equations (5.23) are equivalent to three potential equations,

$$(\Phi^1[u],\Phi^2[u],\Phi^3[u]) = \mathrm{curl}(\tilde{\omega}_1[u],\tilde{\omega}_2[u],\tilde{\omega}_3[u]),$$

which follow from a conservation law $\mathrm{div}(\Phi^1[u],\Phi^2[u],\Phi^3[u])=0$. Since the potential variables $\tilde{\omega}[u]$ are determined to within a gradient of an arbitrary function in $\mathbb{R}^3$, potential equations (5.23) are underdetermined.

Second, suppose $\mathbf{R}\{x,y,z;u\}$ has a conservation law of degree 1 given by (5.16) with corresponding differential form $\omega^{(1)}[U]$ given by expressions (5.15). Using (5.22), one obtains the potential equations,

$$\omega_1[u] = \mathrm{D}_x\tilde{\omega}[u], \quad \omega_2[u] = \mathrm{D}_y\tilde{\omega}[u], \quad \omega_3[u] = \mathrm{D}_z\tilde{\omega}[u], \tag{5.24}$$

i.e., a conservation law $\mathrm{curl}(\omega_1[u],\omega_2[u],\omega_3[u])=0$ yields the potential equations

$$(\omega_1[u],\omega_2[u],\omega_3[u]) = \mathrm{grad}\ \tilde{\omega}[u].$$

Since the potential variable $\tilde{\omega}[u]$ is determined to within a constant, the system of potential equations (5.24) is determined.

2. Potential equations in $\mathbb{R}^4$

Now consider a PDE system $\mathbf{R}\{x,y,z,w;u\}$ in $\mathbb{R}^4$. First, suppose PDE system $\mathbf{R}\{x,y,z,w;u\}$ has a divergence-type (degree three) conservation law (5.14) with corresponding 3-form (5.13). From formula (5.22), it follows that the corresponding four potential equations given by

$$\omega_{234}[u] \equiv \Phi^1[u] = \mathrm{D}_y\tilde{\omega}_{34}[u] - \mathrm{D}_z\tilde{\omega}_{24}[u] + \mathrm{D}_w\tilde{\omega}_{23}[u],$$

$$\omega_{134}[u] \equiv -\Phi^2[u] = \mathrm{D}_x\tilde{\omega}_{34}[u] - \mathrm{D}_z\tilde{\omega}_{14}[u] + \mathrm{D}_w\tilde{\omega}_{13}[u],$$

$$\omega_{124}[u] \equiv \Phi^3[u] = \mathrm{D}_w\tilde{\omega}_{12}[u] - \mathrm{D}_y\tilde{\omega}_{14}[u] + \mathrm{D}_x\tilde{\omega}_{24}[u],$$

$$\omega_{123}[u] \equiv -\Phi^4[u] = \mathrm{D}_x\tilde{\omega}_{23}[u] - \mathrm{D}_y\tilde{\omega}_{13}[u] + \mathrm{D}_z\tilde{\omega}_{12}[u], \tag{5.25}$$

involve six potential variables $\tilde{\omega}_{12}[u]$, $\tilde{\omega}_{13}[u]$, $\tilde{\omega}_{14}[u]$, $\tilde{\omega}_{23}[u]$, $\tilde{\omega}_{24}[u]$, and $\tilde{\omega}_{34}[u]$.

Second, suppose PDE system $\mathbf{R}\{x,y,z,w;u\}$ has a conservation law of degree 2 given by (5.20), with corresponding differential form (5.19). Using (5.22), one obtains six potential equations given by

$$\omega_{12}[u] = \mathrm{D}_x\tilde{\omega}_2[u] - \mathrm{D}_y\tilde{\omega}_1[u], \quad \omega_{13}[u] = \mathrm{D}_x\tilde{\omega}_3[u] - \mathrm{D}_z\tilde{\omega}_1[u],$$

$$\omega_{14}[u] = \mathrm{D}_x\tilde{\omega}_4[u] - \mathrm{D}_w\tilde{\omega}_1[u], \quad \omega_{23}[u] = \mathrm{D}_y\tilde{\omega}_3[u] - \mathrm{D}_z\tilde{\omega}_2[u],$$

$$\omega_{24}[u] = \mathrm{D}_y\tilde{\omega}_4[u] - \mathrm{D}_w\tilde{\omega}_2[u], \quad \omega_{34}[u] = \mathrm{D}_z\tilde{\omega}_4[u] - \mathrm{D}_w\tilde{\omega}_3[u], \tag{5.26}$$

involving four potential variables $\tilde{\omega}_i[u]$, $i=1,2,3,4$.

Finally, suppose PDE system $\mathbf{R}\{x,y,z,w;u\}$ has a conservation law of degree 1, given by (5.18), with corresponding differential form (5.17). The corresponding six potential equations are given by

$$\omega_1[u] = \mathrm{D}_x\tilde{\omega}[u], \quad \omega_2[u] = \mathrm{D}_y\tilde{\omega}[u],$$

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$$\omega_3[u] = \mathrm{D}_z\tilde{\omega}[u], \quad \omega_4[u] = \mathrm{D}_w\tilde{\omega}[u], \tag{5.27}$$

in terms of a single potential variable $\tilde{\omega}[u]$.

3. Potential equations arising from conservation laws of degrees $n-1$ and 1 in $\mathbb{R}^n$

Consider PDE system $\mathbf{R}\{x;u\}$ (3.1) in $\mathbb{R}^n$ with independent variables $x=(x^1,\ldots,x^n)$. First, suppose PDE system $\mathbf{R}\{x;u\}$ has a divergence-type (degree $n-1$) conservation law $\mathrm{D}_\nu\Phi^\nu[u]=0$ with corresponding $n-1$-form given by (5.11). From (5.22), one obtains the corresponding n potential equations,

$$\omega_{\alpha_1\ldots\alpha_r}[u] \equiv \varepsilon_{\nu\alpha_1\ldots\alpha_r}\Phi^\nu[u] = \sum_{i=1}^{n-1}(-1)^{i-1}\frac{\partial}{\partial x^{\alpha_i}}\tilde{\omega}_{\alpha_1\ldots\overline{\alpha_i}\ldots\alpha_{n-1}}[u]. \tag{5.28}$$

Since $\tilde{\omega}_{\alpha_1\ldots\overline{\alpha_i}\ldots\alpha_{n-1}}[u]$ has $n(n-1)/2$ components and is totally antisymmetric, one may relabel $\tilde{\omega}_{\alpha_1\ldots\overline{\alpha_i}\ldots\alpha_{n-1}}=\varepsilon_{\alpha_i\nu\alpha_1\ldots\overline{\alpha_i}\ldots\alpha_{n-1}}v^{\alpha_i\nu}$. After simplification, one obtains the n potential equations ($v^{\nu j}=-v^{j\nu}$) given by

$$\Phi^\nu[u] = \frac{\partial}{\partial x^j}v^{\nu j}, \tag{5.29}$$

which coincide with previously obtained formulas (3.9). The system of potential equations (5.29) is underdetermined. In particular, for an *arbitrary* $r-2$-form $\hat{\omega}^{(r-2)}$, $\mathrm{d}^2\hat{\omega}^{(r-2)}=0$, the potential equations $\omega^{(r)}=\mathrm{d}\tilde{\omega}^{(r-1)}$ imply that the potential variables $\tilde{\omega}^{(r-1)}$ are defined to within arbitrary functions $\mathrm{d}\hat{\omega}^{(r-2)}$. Another way to see this is expressed by gauge freedom (3.10).

Second, suppose PDE system $\mathbf{R}\{x;u\}$ (3.1) has a conservation law of degree 1. The corresponding differential 1-form $\omega^{(1)}[U]$ is given by

$$\omega^{(1)}[U] = \omega_i[U]\mathrm{d}x^i, \tag{5.30}$$

and the conservation law itself is given by the components of the equation $\mathrm{d}\omega^{(1)}[u]=0$, i.e., the $n(n-1)/2$ equations

$$\mathrm{D}_i\omega_j[u] - \mathrm{D}_j\omega_i[u] = 0, \quad i=1,\ldots,n, \quad j=1,\ldots,i-1. \tag{5.31}$$

Using formula (5.22), one obtains the corresponding n potential equations given by

$$\omega_i[u] = \mathrm{D}_i\tilde{\omega}[u], \quad i=1,\ldots,n, \tag{5.32}$$

which are the n components of the relation $\omega^{(1)}=\mathrm{d}\tilde{\omega}^{(0)}$. Indeed, since a zero-form $\tilde{\omega}^{(0)}[u]=\tilde{\omega}[u]$ is a scalar function, potential equations (5.32) are a determined PDE system.

D. Determinedness of sets of potential equations arising from conservation laws of degree r

Consider PDE system $\mathbf{R}\{x;u\}$ (3.1) with $n\geq 2$ independent variables $x=(x^1,\ldots,x^n)$ and $m\geq 1$ dependent variables $u=(u^1,\ldots,u^m)$. Suppose it has a conservation law of degree r, $1\leq r\leq n-1$, and the corresponding potential system $\mathrm{S}\{x;u,\tilde{\omega}\}$ is given by the union of $\mathbf{R}\{x;u\}$ and potential equations (5.22).

It is important to specify for which r the potential system $\mathbf{S}\{x;u,\tilde{\omega}\}$ is determined. From example (5.30)–(5.32), it is easy to see that this is only the case for conservation laws of degree 1, which yield a single potential variable. In particular, the following theorem holds.

Theorem 5.1: *Suppose PDE system $\mathbf{R}\{x;u\}$ (3.1) with $n\geq 2$ independent variables has a conservation law of degree r, $1\leq r\leq n-1$. Then the potential system $\mathbf{S}\{x;u,\tilde{\omega}\}$ given by the union of $\mathbf{R}\{x;u\}$ and potential equations (5.22) is determined if and only if $r=1$.*

The proof is given in Appendix A.

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It follows that for conservation laws of degrees 2 to $n-1$, one must append an appropriate number of gauge constraints to the potential system $\mathbf{S}\{x;u,\tilde{\omega}\}$ in order to make it determined.

E. Direct construction of conservation laws of degree *r*

Now consider the systematic construction of the local conservation laws of degree r, $1\leq r \leq n-1$, for PDE system $\mathbf{R}\{x;u\}$ (3.1) with $n\geq 3$ independent variables $x=(x^1,\ldots,x^n)$ and $m\geq 1$ dependent variables $u=(u^1,\ldots,u^m)$.

The direct construction of divergence-type conservation laws (i.e., $r=n-1$) proceeds by the direct method mentioned in Sec. II A. First, choose a sufficiently general dependence of the conservation law multipliers $\Lambda_\sigma[U]$ on $x,U,\partial U,\ldots$, to some finite order, where $U(x)$ is arbitrary, and accordingly solve the multiplier determining equations which are obtained by the action of each of the Euler differential operators on linear combination (2.4). Second, after finding sets of multipliers, fluxes of each corresponding conservation law can be computed. For details, see Refs. 11, 14, and 29. Theorem 3.1 ensures the completeness of the direct method for PDE systems in solved form. In this section, we therefore only need to consider lower-degree conservation laws $1\leq r<n-1$.

Since any lower-degree conservation law (5.9) is equivalent to a set of $\binom{n}{r+1}$ divergence-type conservation laws given by formula (5.10), the direct method can be applied, i.e., for an arbitrary function $U(x)$, one can seek multipliers $\{\Lambda_\sigma^{(i)}[U]\}$, such that each independent component of $\Omega^{(r+1)}[U]$ is a linear combination,

$$\Omega_{\mu_1\cdots\mu_{r+1}}[U]=\Lambda_\sigma^{(i)}[U]R^\sigma[U], \quad i=1,\ldots,\binom{n}{r+1}. \tag{5.33}$$

The determining equations for the multipliers follow from Definition 5.1. For lower-degree conservation law (5.8) to hold, it is necessary and sufficient that $\mathrm{d}\Omega^{(r+1)}[U]=\mathrm{d}^2\omega^{(r)}[U]=0$ for an *arbitrary function* $U(x)$.

The multiplier determining equations are thus given by the $\binom{n}{r+2}$ vanishing expressions

$$\left(\frac{\partial}{\partial x^\lambda}(\Lambda_\sigma^{(i)}[U]R^\sigma[U])\right)\mathrm{d}x^\lambda\wedge \mathrm{d}x^{\mu_1}\wedge\ldots\wedge \mathrm{d}x^{\mu_{r+1}}\equiv 0. \tag{5.34}$$

Moreover, if the given PDE system is written in a solved form with respect to some leading derivatives, it follows from Theorem 3.1 that for any lower-degree conservation law (5.9), each of its components $\Omega_{\mu_1\cdots\mu_{r+1}}[U]$ can be written in terms of multipliers $\{\Lambda_\sigma^{(i)}[U]\}$ depending on x, U, and sufficiently many derivatives of U.

The following theorem has been established.

Theorem 5.2: (The direct method for construction of lower-degree conservation laws) *For PDE system* $\mathbf{R}\{x;u\}$ *(3.4) written in a solved form with respect to some leading derivatives, all components of each of its conservation laws (5.9) of degree r,* $1\leq r\leq n-2$, *can be represented as linear combinations (5.33) of equations of* $\mathbf{R}\{x;u\}$ *taken with multipliers* $\{\Lambda_\sigma^{(i)}[U]\}=\{\Lambda_\sigma^{(i)}\times(x,U,\partial U,\ldots,\partial^q U)\}$ *for some* $q\geq 0$. *The multipliers satisfy determining Eqs. (5.34) for an arbitrary function* $U=U(x)$.

As a simple example, consider the linear PDE system of Maxwell's equations in a vacuum given by

$$\operatorname{div}\mathbf{B}=0, \quad \operatorname{div}\mathbf{E}=0,$$

$$\frac{\partial}{\partial t}\mathbf{E}=\operatorname{curl}\mathbf{B}, \quad \frac{\partial}{\partial t}\mathbf{B}=-\operatorname{curl}\mathbf{E}, \tag{5.35}$$

where $\mathbf{B}=B^1\mathbf{e}_x+B^2\mathbf{e}_y+B^3\mathbf{e}_z$ is a magnetic field, $\mathbf{E}=E^1\mathbf{e}_x+E^2\mathbf{e}_y+E^3\mathbf{e}_z$ is an electric field, (x,y,z) are Cartesian coordinates, and t is time. Consider a three-dimensional reduction of PDE system

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(5.35) when $\mathbf{B}=b(x,y,t)\mathbf{e}_z$, $\mathbf{E}=e^1(x,y,t)\mathbf{e}_x+e^2(x,y,t)\mathbf{e}_y$. Then Maxwell's equations (5.35) can be written as the PDE system $\mathbf{M}\{t,x,y;e^1,e^2,b\}$ given by

$$R^1[e^1,e^2,b]=e^1_x+e^2_y=0,\quad R^2[e^1,e^2,b]=e^1_t-b_y=0,$$

$$R^3[e^1,e^2,b]=e^2_t+b_x=0,\quad R^4[e^1,e^2,b]=b_t+e^2_x-e^1_y=0. \tag{5.36}$$

We seek a degree one (i.e., curl-type) conservation law of (5.36). Assume a simple but sufficiently general dependence of multipliers $\Lambda^{(i)}_\sigma[E^1,E^2,B]=\Lambda^{(i)}_\sigma[t,x,y,E^1,E^2,B]$, $\sigma=1,\ldots,4$; $i=1,2,3$, such that

$$(\Lambda^{(1)}_\sigma[E^1,E^2,B]R^\sigma[E^1,E^2,B],\Lambda^{(2)}_\sigma[E^1,E^2,B]R^\sigma[E^1,E^2,B],\Lambda^{(3)}_\sigma[E^1,E^2,B]R^\sigma[E^1,E^2,B])$$
$$=\mathrm{curl}_{(t,x,y)}\; W[E^1,E^2,B]$$

for some vector W in terms of arbitrary functions E^1, E^2, B and their derivatives.

The determining equations for the multipliers are given by div(curl $W[E^1,E^2,B])\equiv 0$. Setting to zero the coefficients of all derivatives of E^1, E^2, B in the determining equations, one finds

$$\Lambda^{(i)}_\sigma=C\delta^i_\sigma=\begin{cases}C, & i=\sigma,\\ 0, & i\neq\sigma,\end{cases}\qquad C=\text{const.}$$

This yields $W[E^1,E^2,B]=(B,-E^2,E^1)$, and the resulting curl-type conservation law is given by

$$\mathrm{curl}_{(t,x,y)}\; W[e^1,e^2,b]\equiv(e^1_x+e^2_y,e^1_t-b_y,e^2_t+b_x)=0 \tag{5.37}$$

of PDE system (5.36).

The above example has the following physical interpretation. Let $\eta^{ij}=\eta_{ij}=\mathrm{diag}(-1,1,1)$ be the standard space-time metric tensor in three-dimensional Minkowski space with coordinates $(x^0,x^1,x^2)=(t,x,y)$. A skew-symmetric electromagnetic field tensor F_{ij} and its raised version $F^{ij}=\eta^{jk}\eta^{jl}F_{kl}$ are, respectively, given by the matrices

$$F_{ij}=\begin{pmatrix}0 & -e^1 & -e^2\\ e^1 & 0 & b\\ e^2 & -b & 0\end{pmatrix},\quad F^{ij}=\begin{pmatrix}0 & e^1 & e^2\\ -e^1 & 0 & b\\ -e^2 & -b & 0\end{pmatrix}. \tag{5.38}$$

In (5.38), the indices i, j take on the values 0, 1, 2, with 0 corresponding to time and 1, 2 to x- and y-components, respectively.

The electromagnetic tensor F_{ij} corresponds to a 2-form,

$$F=-e^1\mathrm{d}t\wedge\mathrm{d}x-e^2\mathrm{d}t\wedge\mathrm{d}y+b\mathrm{d}x\wedge\mathrm{d}y.$$

The dual of F_{ij} is the tensor given by $*F_k=\frac{1}{2}\varepsilon_{ijk}F^{ij}$, where ε_{ijk} is the Levi–Civita symbol. The dual tensor $*F$ corresponds to a 1-form given by

$$*F=bdt-e^2dx+e^1dy.$$

As is well-known, using the differential forms F and $*F$, one can express Maxwell's equations (5.36) in the elegant form

$$\mathrm{d}F=0,\quad \mathrm{d}*F=0, \tag{5.39}$$

where d is the exterior derivative. Note that in (5.39), the equation $\mathrm{d}F=0$ is equivalent to the scalar equation $R^4[e^1,e^2,b]=0$, while the equation and $\mathrm{d}*F=0$ is equivalent to the remaining three scalar equations of (5.36). Consequently, Eqs. (5.39) can be formally written as the conservation laws,

$$\mathrm{div}_{(t,x,y)}[b,e^2,-e^1]=0,\quad \mathrm{curl}_{(t,x,y)}[b,-e^2,e^1]=0. \tag{5.40}$$

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VI. NONLOCAL SYMMETRIES AND NONLOCAL CONSERVATION LAWS OF MULTIDIMENSIONAL PDE SYSTEMS

Consider PDE system $\mathbf{R}\{x;u\}$ (3.1) with $n>2$ independent variables $x=(x^1,\ldots,x^n)$ and $m\geq 1$ dependent variables $u=(u^1,\ldots,u^m)$.

Suppose one finds $q\geq 1$ local conservation laws of $\mathbf{R}\{x;u\}$, which may be divergence-type or lower-degree conservation laws. For each conservation law, through potential equations [given by (3.9) or (5.22), respectively], one can introduce potential variables labeled by $v^{(i)}$, $i=1,\ldots,q$. (A potential variable $v^{(i)}$ is a scalar if it arises from a conservation law of degree $r=1$, and a vector otherwise.)

Through the introduced potential variables, one can construct q singlet potential systems $\mathbf{S}^{(1)}\{x;u,v^{(i)}\}$, $\binom{q}{2}$ couplets: $\mathbf{S}^{(2)}\{x;u,v^{(i)},v^{(j)}\}$, $\binom{q}{3}$ triplets: $\mathbf{S}^{(3)}\{x;u,v^{(i)},v^{(j)},v^{(l)}\},\ldots$, one q-plet: $\mathbf{S}^{(q)}\{x;u,v^{(1)},\ldots,v^{(q)}\}$, which taken all together form a *combination potential system* $\mathbb{P}_{v^1\ldots v^q}$.

For each such potential system, the following questions are of importance.

(1) Is it possible to use a potential system to obtain nonlocal symmetries of the given PDE system $\mathbf{R}\{x;u\}$?
(2) Through application of the direct method to a potential system, can one find additional (nonlocal) conservation laws of the given PDE system $\mathbf{R}\{x;u\}$?

These questions are considered in the following subsections.

A. Nonlocal symmetries

Similar to the two-dimensional situation (Sec. II F), a *nonlocal symmetry* of a multidimensional PDE system $\mathbf{R}\{x;u\}$ is a symmetry of its solution manifold which is not a local symmetry of $\mathbf{R}\{x;u\}$, when acting on its local space of variables $(x,u,\partial u,\ldots)$.

It turns out that it is only possible to compute a nonlocal symmetry of $\mathbf{R}\{x;u\}$, arising from a local symmetry of a potential system of $\mathbf{R}\{x;u\}$, provided the potential system is a determined system. In particular, the following essential theorem holds.[11]

Theorem 6.1: *Each local symmetry of an underdetermined potential system* $\mathbf{S}\{x;u,v\}$ *projects onto a local symmetry of a given system* $\mathbf{R}\{x;u\}$ *(3.1).*

A natural question arises: for a given PDE system $\mathbf{R}\{x;u\}$ (3.1) with a divergence-type conservation law, how does one choose gauges (i.e., additional algebraic or differential equations relating potentials) which, when appended to an underdetermined potential system $\mathbf{S}\{x;u,v\}$, yield nonlocal symmetries of a PDE system $\mathbf{R}\{x;u\}$? The general answer to this question is not known. However, for some PDE systems, it has been demonstrated that there are gauges that lead to finding nonlocal symmetries.

Examples of nonlocal symmetries computed using a Lorentz gauge appeared in Refs. 11 and 12 (see also Sec. 5.3 of Ref. 14). These examples as well as new examples using a divergence gauge are exhibited in the companion paper.[1]

Remark 6.1: Since nonlocally related subsystems in multidimensions are always determined, they provide a natural framework for seeking nonlocal symmetries of a given system. New examples of nonlocal symmetries arising from nonlocally related subsystems are provided in Ref. 1.

B. Nonlocal conservation laws

Several examples of nonlocal conservation laws of a given PDE system that arise as local conservation laws of a potential system of degree r are known, in particular, for the linear wave equation, and the linear Maxwell's equations in $(2+1)$ and $(3+1)$ dimensions.[12] These examples are reviewed in Ref. 1. In particular, the following theorem holds.[12]

Theorem 6.2: *Suppose a given PDE system* $\mathbf{R}\{x;u\}$ *has an underdetermined potential system* $\mathbf{S}\{x;u,v\}$ *with gauge freedom given by point symmetry* X_{gauge} *(3.11). Then all divergence-type conservation laws,*

$$\operatorname{div}\Phi[u,v]=\mathrm{D}_i\Phi^i[u,v]=0,$$

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of $\mathbf{S}\{x;u,v\}$ *are gauge-invariant under (3.11). In particular,* $\operatorname{div}(\mathrm{X}_{gauge}\Phi)\equiv 0$ *on solutions of* $\mathbf{S}\{x;u,v\}$.

Theorem 6.2 states that fluxes are invariant under gauge symmetries but does not rule out the possible explicit dependence of fluxes $\Phi^i[u,v]$ on potentials v. Examples are known of nonlocal conservation laws that arise from both determined and underdetermined potential systems.[12]

Construction of nonlocal conservation laws. For PDE systems with two independent variables, in Theorem 2.2 it has been shown that to construct nonlocal conservation laws from a potential system, multipliers must have an essential dependence on potential variables. We now generalize Theorem 2.2 to the multidimensional situation.

Theorem 6.3: *Suppose* $\mathbf{R}\{x;u\}$ *(3.1) is a PDE system for which* $K\geq 1$ *local divergence-type conservation laws (3.2) are known. Let* $\mathbf{S}\{x;u,v\}$ *be the potential system given by the union of* $\mathbf{R}\{x;u\}$ *and the corresponding K systems of potential equations (3.9). Then each divergence-type conservation law of the potential system* $\mathbf{S}\{x;u,v\}$*, arising from multipliers that do not depend on potential variables v, is linearly dependent on local conservation laws of* $\mathbf{R}\{x;u\}$.

The proof is given in Appendix B.

We conjecture that the statement of Theorem 6.3 holds, in general, i.e., for the construction of nonlocal conservation laws of degree r arising from a potential system of degree q, $1\leq r,q\leq n-1$.

We also remark that in practice, the application of the direct conservation law construction procedure to potential systems can be useful whether a newly obtained conservation law is local or nonlocal, as long as such a conservation law is new, i.e., linearly independent of previously known local conservation laws of a given PDE system.

VII. SYSTEMATIC CONSTRUCTION OF NONLOCALLY RELATED MULTIDIMENSIONAL PDE SYSTEMS

Now consider the problem of the construction of PDE systems nonlocally related to a given PDE system $\mathbf{R}\{x;u\}$ (3.1) with $n\geq 3$ independent variables.

We outline a systematic procedure for constructing a hierarchy (tree) of nonlocally related potential systems and subsystems, which generalizes the situation for two-dimensional PDE systems (Sec. II E) through inclusion of lower-degree conservation laws.

(1) *Construction of local conservation laws.* Use the direct method (Sec. V E) to find a set $\{\mathcal{K}_i\}$ of linearly independent and inequivalent local conservation laws of degrees $r=1,\ldots,n-1$ for the PDE system $\mathbf{R}\{x;u\}$. Let q be the number of such conservation laws that are found.

(2) *Construction of potential systems.* Use the set of known local conservation laws $\{\mathcal{K}_i\}$ to introduce q sets of potential variables $v^{(i)}$, $i=1,\ldots,q$, using potential equations (3.9) (for divergence-type conservation laws) and (5.22) (for lower-degree conservation laws). Construct the corresponding combination potential system $\mathbb{P}_{v^{(1)}\ldots v^{(q)}}$ which contains 2^q-1 potential systems. Together with the given PDE system $\mathbf{R}\{x;u\}$, this yields a tree $\mathcal{T}_1$ with up to 2^q nonlocally related systems.

(3) *Additional conservation laws.* In the tree $\mathcal{T}_1$, consider the q-plet potential system, $\mathbf{S}^{(M)}\{x;u,v^{(1)},\ldots,v^{(q)}\}$. For this q-plet, seek linearly independent conservation laws. (If using the direct method, normally seek multipliers that have an essential dependence on the potential variables $v^{(1)},\ldots,v^{(q)}$.) Eliminate conservation laws that are linearly dependent on the set of known local conservation laws $\{\mathcal{K}_i\}$ of $\mathbf{R}\{x;u\}$. Let q' be the number of newly obtained linearly independent conservation laws of $\mathbf{S}^{(q)}\{x;u,v^{(1)},\ldots,v^{(q)}\}$. Introduce corresponding sets of potential variables $v^{(j)}$, $j=q+1,\ldots,q+q'$. (By construction, the full set of potentials $\{v^{(1)},\ldots,v^{(q+q')}\}$ is linearly independent.)

(4) *Tree extension.* Use the $q+q'$ sets of potential variables $\{v^{(i)}\}$ to construct the corresponding combination potential system $\mathbb{P}_{v^{(1)}\ldots v^{(M+M')}}$. Together with the given system $\mathbf{R}\{x;u\}$, this yields an extended tree $\mathcal{T}_2$.

(5) *Continuation.* Repeat steps (3) and (4) for the tree $\mathcal{T}_2$, until no further linearly independent conservation laws are found for any nonlocally related potential system. This yields a possibly larger extended tree $\mathcal{T}_3$.

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(6) *Construction of subsystems.* For all systems in the tree $\mathcal{T}_3$, exclude where possible, one by one, dependent variables, to generate subsystems of the systems in the tree $\mathcal{T}_3$. Eliminate locally related subsystems. In addition, in the same manner generate nonlocally related subsystems obtained after an interchange of one or more independent and dependent variables. This yields a possibly larger extended tree of nonlocally related systems denoted by $\mathcal{T}_4$.

As a simple example, consider the PDE system $\mathbf{M}\{t,x,y;e^1,e^2,b\}$ for Maxwell's equations in $(2+1)$ dimensions, given by (5.36), which is equivalent to the set of conservation laws (5.40). The first conservation law of (5.40) yields the potential equations $\mathcal{P}^{(1)}$ given by

$$b = a^2_x - a^1_y,$$

$$e^2 = a^0_y - a^2_t,$$

$$-e^1 = a^1_t - a^0_x \tag{7.1}$$

in terms of a vector potential variable $a=(a^0,a^1,a^2)$, and the second conservation law of (5.40) yields the potential equations $\mathcal{P}^{(2)}$ given by

$$b = w_t,$$

$$-e^2 = w_x,$$

$$e^1 = w_y \tag{7.2}$$

in terms of a scalar potential variable w. As a result, one obtains potential systems $\mathbf{MA}\{t,x,y;b,e^1,e^2,a\}=\mathbf{M}\{t,x,y;b,e^1,e^2\}\cup\mathcal{P}^{(1)}$ (underdetermined) and $\mathbf{MW}\{t,x,y;b,e^1,e^2,w\}$ $=\mathbf{M}\{t,x,y;b,e^1,e^2\}\cup\mathcal{P}^{(2)}$ (determined). A couplet potential system $\mathbf{MAW}\{t,x,y;a,w\}$ is given by

$$w_t = a^2_x - a^1_y,$$

$$-w_x = a^0_y - a^2_t,$$

$$-w_y = a^1_t - a^0_x,$$

$$a^0_t - a^1_x - a^2_y = 0, \tag{7.3}$$

where the components of the electric and magnetic fields are excluded through obvious substitutions.

A nonlocally related subsystem $\mathbf{E}\{t,x,y;e^1,e^2\}$ is obtained directly from Maxwell's equations (5.36) by eliminating the magnetic field b,

$$e^1_x + e^2_y = 0,$$

$$e^1_{tt} = e^1_{xx} + e^1_{yy},$$

$$e^2_{tt} = e^2_{xx} + e^2_{yy}. \tag{7.4}$$

In summary, a tree

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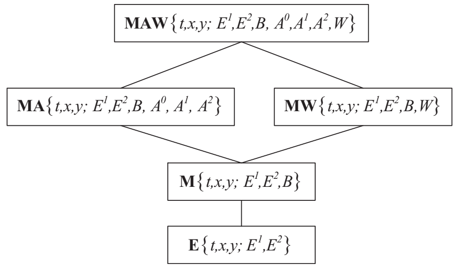

FIG. 1. A tree of nonlocally related systems for Maxwell's equations (5.36) in 3D Minkowski space.

$$\mathcal{T}_M = \{\mathbf{M}\{t,x,y;b,e^1,e^2\},\mathbf{MW}\{t,x,y;b,e^1,e^2,w\},\mathbf{MA}\{t,x,y;b,e^1,e^2,a\},\mathbf{MAW}\{t,x,y;a,w\},\mathbf{E}\{t,x,y;e^1,e^2\}\} \tag{7.5}$$

of nonlocally related PDE systems has been constructed for Maxwell's equation (5.36) in three-dimensional Minkowski space. This tree is presented in Fig. 1.

Tree $\mathcal{T}_M$ (7.5) can be further extended by considering each of the equations $R^\sigma[e^1,e^2,b]=0$ in (5.36), $\sigma=1,2,3$, as additional divergence-type conservation laws. This leads to additional systems of potential equations given by

$$\mathcal{P}^{(3)}:\begin{cases}0 = p_x^2 - p_y^1\\ e^1 = p_y^0 - p_t^2\\ e^2 = p_t^1 - p_x^0,\end{cases} \qquad \mathcal{P}^{(4)}:\begin{cases}e^1 = q_x^2 - q_y^1\\ 0 = q_y^0 - q_t^2\\ -b = q_t^1 - q_x^0,\end{cases} \qquad \mathcal{P}^{(5)}:\begin{cases}e^2 = r_x^2 - r_y^1\\ b = r_y^0 - r_t^2\\ 0 = r_t^1 - r_x^0,\end{cases} \tag{7.6}$$

involving vector potentials $p=(p^0,p^1,p^2)$, $q=(q^0,q^1,q^2)$, $r=(r^0,r^1,r^2)$.

As a result, one obtains an extended tree $\mathcal{T}_M^{(2)}$ including the following 33 nonlocally related PDE systems.

- The given PDE system $\mathbf{M}\{t,x,y;b,e^1,e^2\}$.
- The nonlocally related electric field subsystem $\mathbf{E}\{t,x,y;e^1,e^2\}$.
- Five singlet potential systems $\mathbf{MW}\{t,x,y;b,e^1,e^2,w\}$, $\mathbf{MA}\{t,x,y;b,e^1,e^2,a\}$, $\mathbf{MP}\{t,x,y;b,e^1,e^2,p\}$, $\mathbf{MQ}\{t,x,y;b,e^1,e^2,q\}$, $\mathbf{MR}\{t,x,y;b,e^1,e^2,r\}$.
- Ten couplet potential systems involving pairs of the potential variables w,a,p,q,r.
- Ten triplet potential systems involving three of the potential variables w,a,p,q,r.
- Five quadruplet potential systems.
- The 5-plet potential system $\mathbf{MWAPQR}\{t,x,y;b,e^1,e^2,w,a,p,q,r\}$.

(In each potential system in the tree $\mathcal{T}_M^{(2)}$, where applicable, one or more of the original variables b,e^1,e^2 may be kept or substituted in terms of potential variables, if redundant.)

In the companion paper,[1] we illustrate the use of the determined potential system $\mathbf{MW}\{t,x,y;b,e^1,e^2,w\}$ and the underdetermined potential system $\mathbf{MAW}\{t,x,y;a,w\}$ (appended with a Lorentz gauge constraint) to find nonlocal symmetries for Maxwell's equations (5.36).[11]

VIII. DISCUSSION

The current paper discusses the construction of nonlocally related PDE systems with $n\geq 3$ independent variables. For a given system, nonlocally related systems can arise as (i) nonlocally related subsystems obtained by exclusion of variables (Sec. IV) and/or as (ii) potential systems arising from local conservation laws.

For a PDE system with n independent variables, one can have several types of local conservation laws: divergence-type conservation laws (degree $r=n-1$) and lower-degree conservation laws (degrees $1\leq r<n-1$). All such conservation laws can be sought through a systematic procedure (direct construction method) involving multipliers, as explained in the corresponding sections.

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Both divergence-type and lower-degree local conservation laws can be used to generate potential equations. For conservation laws of degree 1, a scalar potential variable arises, and the potential system is determined. For all other types of conservation laws (degrees $1<r\leq n-1$), potential equations involve several variables and are underdetermined (i.e., have an infinite number of gauge symmetries). (For PDE systems with two independent variables, the situation is much simpler. Here one only has divergence type conservation laws of degree $r=n-1=1$, and the resulting potential equations are determined.)

Applications of nonlocally related PDE systems include the systematic construction of nonlocal symmetries and nonlocal conservation laws of a given PDE system. Such symmetries and conservation laws do not arise from standard local procedures directly applied to a given system, but can arise from the same standard local procedures applied to a nonlocally related system. Nonlocal symmetries can only arise from determined nonlocally related PDE systems (nonlocally related subsystems and determined potential systems). In order to seek nonlocal symmetries of a given PDE system, underdetermined potential systems have to be appended with gauge constraints (Sec. III).

In order to construct nonlocal conservation laws of a given PDE system, one can apply the direct construction method to any potential system. If it is computationally feasible, it makes sense to work with the potential system with the largest number of potentials. (It generally does not make sense to apply the direct construction method to a nonlocally related subsystem.) It is important to note that unlike the situation for obtaining nonlocal symmetries, nonlocal conservation laws can be obtained from underdetermined potential systems *without gauge constraints*, as it is shown for Maxwell's equations in the accompanying paper.[1]

It is conjectured that in order to obtain nonlocal conservation laws (of any degree) from potential systems (of any degree) using the direct method, it is necessary to use multipliers that essentially depend on nonlocal variables. This has been proven for the case of obtaining nonlocal divergence-type conservation laws from potential systems of degree $n-1$ (i.e., potential systems following from divergence-type conservation laws).

Finally, a systematic procedure for the construction of PDE systems nonlocally related to a given one in multidimensions was presented and illustrated.

Examples pertaining to the current paper are presented in the accompanying paper.[1] These include new and known examples involving the construction and application of nonlocally related PDE systems in $n\geq 3$ dimensions.

Important open problems include the following.

(1) For a given underdetermined potential system (especially, in the case of a nonlinear PDE system), how does one choose gauge constraints so that nonlocal symmetries can be effectively computed? Even for simple PDE systems with $n\geq 3$ independent variables, so far it has not been computationally feasible to perform a sufficiently general classification of gauge constraints that can yield nonlocal symmetries.

(2) Is it possible to determine in advance, which potential system(s) of a given PDE system are more likely to yield nonlocal symmetries and nonlocal conservation laws?

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APPENDIX A: PROOF OF THEOREM 5.1

Proof: If $r=1$, the corresponding potential equations are given by (5.32), and the only potential variable $\tilde{\omega}[u]$ is obviously determined to within a constant.

If $r>1$, the argument is as follows. Let the fluxes of the conservation law of degree r be given by an r-form $\omega^{(r)}[U]$, and the conservation law equations by the components of $\mathrm{d}\omega^{(r)}[u]=0$. The corresponding potential equations are then given by the equations

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$$\omega^{(r)}[u] = \mathrm{d}\widetilde{\omega}^{(r-1)}[u] \tag{A1}$$

for some $\widetilde{\omega}^{(r-1)}[u]$, i.e., by Eqs. (5.21). Since for any differential form $\hat{\omega}^{(q)}$, $\mathrm{d}^2\hat{\omega}^{(q)} \equiv 0$, one can write (A1) as

$$\omega^{(r)}[u] = \mathrm{d}(\widetilde{\omega}^{(r-1)}[u] + \mathrm{d}\hat{\omega}^{(r-2)}(x)),$$

where $\hat{\omega}^{(r-2)}(x)$ is an arbitrary $r-2$ form. It follows that potential variables (components of $\widetilde{\omega}^{(r-1)}[u]$) are defined to within components of $\mathrm{d}\hat{\omega}^{(r-2)}(x)$, which are arbitrary functions of x, and hence potential equations (A1) are underdetermined by definition. ■

APPENDIX B: PROOF OF THEOREM 6.3

Proof: For simplicity, consider a singlet potential system $\mathbf{S}\{x;u,v\}$ following from some divergence-type conservation law (3.2) of PDE system $\mathbf{R}\{x;u\}$ (3.1). (The proof directly carries over to K-plet potential systems, $K>1$.)

Consider a local conservation law,

$$\mathrm{D}_k F^k[u,v] = 0, \tag{B1}$$

of the potential system $\mathbf{S}\{x;u,v\}$, arising from multipliers independent of the potential variables v. Conservation law (B1) corresponds to the divergence expression

$$\mathrm{D}_k F^k[U,V] = A_i[U](\mathrm{D}_j V^{ij} - \Phi^i[U]) + \sum_{\sigma=1}^{N'} \Lambda_\sigma[U]R^\sigma[U], \tag{B2}$$

where $A_i[U]$ are multipliers of the potential equations, and $\Lambda_\sigma[U]$ are multipliers of the equations of $\mathbf{R}\{x;u\}$ (3.1) that are present in the potential system $\mathbf{S}\{x;u,v\}$. (Without loss of generality, summation in σ can be taken from 1 to N.)

Now apply Euler operators $\mathbf{E}_{V^{\alpha\beta}}$ (2.5) with respect to $V^{\alpha\beta}$ ($\alpha,\beta = 1,\ldots,n$, $\alpha \neq \beta$) to Eq. (B2). The divergence expression on the left-hand side and the terms on the right-hand side not involving $V^{\alpha\beta}$ vanish identically. Using the antisymmetry of V^{ij}, one obtains

$$\frac{\partial A_\alpha[U]}{\partial x^\beta} - \frac{\partial A_\beta[U]}{\partial x^\alpha} \equiv 0, \quad \alpha,\beta = 1,\ldots,n.$$

From a basic lemma in variational calculus,[28] it follows that

$$A_i[U] = \frac{\partial B[U]}{\partial x^i}.$$

Hence Eq. (B2) can be rewritten as

$$\mathrm{D}_i(F^i[U,V] - B[U](\mathrm{D}_j V^{ij} - \Phi^i[U])) = B[U]\mathrm{D}_i\Phi^i[U] + \Lambda_\sigma[U]R^\sigma[U]. \tag{B3}$$

(Here the identity $\mathrm{D}_i\mathrm{D}_j V^{ij} \equiv 0$ has been used.)

Now consider the conservation law $\mathrm{D}_i\Phi^i[u]=0$ (3.2). Assuming that $\mathbf{R}\{x;u\}$ (3.1) can be written in solved form (3.4) with respect to some leading derivatives $\{u^{j_\sigma}_{i_{\sigma,1}\cdots i_{\sigma,s}}\}$, it follows that by adding trivial fluxes (of the first type), one can assume that each flux $\Phi^i[U]$ contains no leading derivatives nor their differential consequences. Thus, the only leading derivatives (and their differential consequences) that can arise in the expression $\mathrm{D}_i\Phi^i[U]$ will come from subleading derivatives $\{U^{j_\sigma}_{i_{\sigma,1}\cdots i_{\sigma,s-1}}\}$ (and their differential consequences). Hence, the divergence expression $\mathrm{D}_i\Phi^i[U]$ must be a linear combination of $R^\sigma[U]$ and their differential consequences,

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$$\mathrm{D}_i\Phi^i[U] = \Gamma_{(0)\sigma}[U]R^\sigma[U] + \Gamma^i_{(1)\sigma}[U]\mathrm{D}_iR^\sigma[U] + \cdots + \Gamma^{i_1\ldots i_q}_{(q)\sigma}[U]\mathrm{D}_{i_1}\ldots\mathrm{D}_{i_q}R^\sigma[U],$$

for some coefficients $\Gamma_{(0)\sigma}[U],\ldots,\Gamma^{i_1\ldots i_q}_{(q)\sigma}[U]$ which are nonsingular functions of x,U and derivatives of U. Then one has

$$B[U]\mathrm{D}_i\Phi^i[U] = \Gamma_\sigma[U]R^\sigma[U] + \mathrm{D}_iQ^i[U],$$

where $\{Q^i[U]\}_{i=1}^n$ are linear combinations involving $R^\sigma[U]$ and its differential consequences. Moreover, each $Q^i[U]$ vanishes on all solutions $U(x)=u(x)$ of $\mathbf{R}\{x;u\}$. Therefore, (B3) becomes

$$\mathrm{D}_i(F^i[U,V] - B[U](\mathrm{D}_jV^{ij} - \Phi^i[U]) - Q^i[U]) = (\Lambda_\sigma[U] + \Gamma_\sigma[U])R^\sigma[U]. \tag{B4}$$

Since expressions $B[U](\mathrm{D}_jV^{ij}-\Phi^i[U])$ and $Q^i[U]$ vanish on solutions $(U,V)=(u,v)$ of the potential system $\mathbf{S}\{x;u,v\}$, the left-hand side of (B4) is a divergence expression corresponding to a conservation law equivalent to conservation law (B1). The right-hand side of (B4) is a linear combination of equations of given system $\mathbf{R}\{x;u\}$ (3.1), with multipliers depending only on local variables of $\mathbf{R}\{x;u\}$. Therefore, conservation law (B1) of the potential system $\mathbf{S}\{x;u,v\}$ is equivalent to a local conservation law of the given system $\mathbf{R}\{x;u\}$. ■

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