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Higher-order symmetries and conservation laws of the *G*-equation for premixed combustion and resulting numerical schemes

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Abstract It is shown that the set of computable local symmetries of the *G*-equation for flame-front propagation of premixed combustion is considerably extended if higher-order symmetries are considered. Classical point symmetries are exhaustively discussed by Oberlack et al. (Combust Theor Model 5:363–383, 2001). Further, if the flow velocity is zero, an infinite series of higher-order symmetries has been derived by Oberlack (J Calcutta Math Soc 1:41–52, 2004). Presently it is evidenced that the *G*-equation also admits an infinite number of higher-order symmetries for an arbitrary velocity field. Higher-order symmetries involving derivatives up to second order are computed. Geometrical and kinematic interpretations of the symmetries are given. For the special case of constant flow velocity, an infinite set of local conservation laws of the *G*-equation has been derived using the direct method. It is demonstrated how the derived infinite sets of local symmetries and conservation laws can be used to develop novel numerical schemes (including higher-order ones) to perform computations in practical applications involving the *G*-equation.

Keywords Conservation law · *G*-equation · Level-set equation · Premixed combustion · Symmetry

1 Introduction

In recent years, the *G*-equation for premixed combustion (first derived in [1, pp. 97–131]) has become one of the dominant approaches for modelling premixed combustion in a very broad range of practical applications, including, for example, spark-ignition engines.

The *G*-equation is given by

$$\frac{\partial G}{\partial t} + u_k \frac{\partial G}{\partial x_k} = s_l \sqrt{\frac{\partial G}{\partial x_k}\frac{\partial G}{\partial x_k}}, \tag{1}$$

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where t and $\boldsymbol{x} = (x_1, \ldots, x_N) \in \mathbb{R}^N$ and t are time and space variables, respectively, $\boldsymbol{u} = (u_1, \ldots, u_N)$ is the flow velocity vector of the premixed gaseous mixture of fuel and oxidizer undergoing combustion, $N = 2$ or 3; s_l is the laminar burning velocity, and G denotes a scalar field quantity determining an instantaneous flame front at $G = G_0$. Hence G_0 defines a level-set and Eq. 1 falls into the class of level-set equations. The G field has a physical meaning only at G_0. In (1) and below, summation in repeated indices is assumed. Here, and in all subsequent sections, except for parts of Sect. 3 where it is explicitly discussed, we assume $s_l = \text{const}$, although in some refined models in the literature, $s_l \neq \text{const}$ is also employed. The most important physical effects are curvature and flame stretch which in the limit of thin flames are modelled, e.g. in [2] or [3]. An additional modification of s_l is given in [4] and [5] where the induced velocity arising from heat release is modelled by a point source at the flame front assuming potential flow.

A large amount of applied works have been dedicated to the G-equation (1). In order to make the G-equation amenable to numerical computations, a diversity of numerical schemes have been developed, e.g., [6–10]. All of these are essentially two-step algorithms. In the first step the G-equation is advanced in time for the entire domain of integration. Performing this for a large number of time steps may lead to an unstable computation, either because cusps or steep gradients may develop. Hence, after one or several time steps of advancing the G-equation, a second step is needed to regularize the G-field apart from the front G_0 in order to avoid the above-mentioned numerical problems. In most applications, the distance function equation $|\nabla G| = 1$ is solved, which leads to a smooth unique G-field that is sufficiently regular for the next time step. This regularization step is computationally expensive compared to the advancement of the G-field itself. For an overview of numerical schemes, see [11,12].

Finally, to make the G-approach applicable to the computation of turbulent premixed combustion, a variety of semi-empirical model equations have been proposed, e.g., [13–17] to overcome the disproportional amount of computational resources needed for the computation of the smallest length scales in the problem.

In contrast, the mathematical properties of the G-equation (1) received considerably less attention. In particular, only recently have the important symmetry properties of the G-equation been explored. In [18], classical point symmetries of the G-equation in combination with the equations of fluid dynamics have been computed. It was shown that one particular symmetry, named "generalized scaling symmetry" (relabelling symmetry) has important implications for the understanding and modelling of the G-equation in turbulent flows. New physically sound modelling routes have been suggested.

Higher-order symmetries of the G-equation (1) were first studied in [19] considering the restricted case of no underlying flow velocity, i.e., the flame-front propagation is purely due to the burning process. A remarkably large set of symmetries has been derived for this equation. In terms of Lie point symmetries, it was shown that in $\mathbb{R}^3$, the G-equation (1) has a 16-dimensional Lie algebra of point symmetries plus an infinite family of relabelling symmetries. Extending further to higher-order symmetries, an additional infinite set of symmetries has been found, including infinitesimals with dependence on derivatives of G to arbitrary order.

In the first part of the next Sect. 2, we study Lie point symmetries and higher-order symmetries of the G-equation (1), extending the work of [19] to the case of arbitrary $\boldsymbol{u} \neq 0$.

In Sect. 3 of the present paper we consider the conservation properties of the G-equation (1). Recall that the original derivation of the G-equation is exclusively based on the kinematic balance equation

$$\frac{\mathrm{d}\boldsymbol{x}_f}{\mathrm{d}t} = \boldsymbol{u}(\boldsymbol{x}_f, t) + s_l \boldsymbol{n}, \tag{2}$$

involving the flame-front location $\boldsymbol{x}_f$, the propelling velocities $\boldsymbol{u}$ and $s_l \boldsymbol{n}$; here the first one is the flow velocity and the second one is a product of the laminar burning velocity s_l and the vector $\boldsymbol{n}$ normal to flame surface. Equation (2) has no direct link to any classical conservation law. To the best of the authors' knowledge, no conservation laws or related Casimirs are known for the G-equation.

Within Sect. 3, we systematically seek local divergence-type conservation laws of the G-equation (1), using the recently developed *direct method* [20,21] (see also [22]). The direct method involves seeking multipliers such that a linear combination of the equations of a given system of partial differential equations (PDE) taken with these multipliers yield a divergence expression. The direct method is rigorous, in the sense that, if a given PDE system

can be written in a solved form for some leading derivatives, the method yields all its local conservation laws, with density and fluxes depending on derivatives up to any prescribed order.

In particular, we show that in two and three space dimensions the G-equation (1) with $\boldsymbol{u} = \text{const}$ admits an infinite number of second-order conservation laws, both for the two- and three-dimensional cases. Conservation laws of higher orders have not been considered due to the computational complexity.

As is well-known, conservation laws are important since they provide deep insights into the underlying physics. They are also used in PDE analysis, such as study of existence, uniqueness and stability of solutions. Moreover, conservation laws have the rather pleasant property of allowing the application of a large number of robust and efficient numerical schemes which have been derived in the last 50 years, such as finite-volume, finite-element, or discontinuous Galerkin methods, to name only a few.

In Sect. 4 of the present paper, we discuss applications of the newly derived conservation laws and symmetries of the G-Equation (1) to the development of new numerical algorithms. The first method advances a set of conservation equations which after each time step computes G and all spatial derivatives up to order two. The second method, although not based on conservation laws, also computes a system of equations where each element is a geometrical term of increasing order starting with the normal vector, the Hessian of G and so forth. In both cases an accurate front-reconstruction scheme may be applied to G and its higher-order derivatives after each time step.

2 Local symmetry groups of the G-equation

In general, a *symmetry* of a PDE system is any transformation of its dependent and independent variables that leaves the solution manifold invariant. Therefore, in principle, any PDE system has symmetries. The simplest discrete and continuous symmetries, such as reflections and translations, can often be found by inspection. One can also systematically seek *Lie point symmetries*, i.e., point symmetries that form one-parameter Lie groups. For the G-equation (1), such symmetries have the global form

$$t^* = f(t, \boldsymbol{x}, G; \varepsilon), \quad \boldsymbol{x}^* = \mathbf{g}(t, \boldsymbol{x}, G; \varepsilon), \quad G^* = h(t, \boldsymbol{x}, G; \varepsilon), \tag{3}$$

where ε is the group parameter. Locally, symmetries (3) have the form [22, Chap. 1; 23, Chap. 2]

$$\begin{aligned} t^* &= t + \varepsilon \xi^t(t, \boldsymbol{x}, G) + O(\varepsilon^2), \\ x_i^* &= x_i + \varepsilon \xi_i^x(t, \boldsymbol{x}, G) + O(\varepsilon^2), \quad i = 1, \ldots, n, \\ G^* &= G + \varepsilon \eta(t, \boldsymbol{x}, G) + O(\varepsilon^2), \end{aligned} \tag{4}$$

where the infinitesimal components are given by

$$\xi^t = \frac{\partial f}{\partial \varepsilon}\Big|_{\varepsilon=0}, \quad \xi_i^x = \frac{\partial g_i}{\partial \varepsilon}\Big|_{\varepsilon=0}, \quad \eta = \frac{\partial h}{\partial \varepsilon}\Big|_{\varepsilon=0}. \tag{5}$$

The upper indices exponents of the infinitesimal components denote the variables they refer to.

To find point symmetries of the G-equation in (1), one considers symmetry generators,

$$\mathrm{X} = \xi^t \frac{\partial}{\partial t} + \xi_i^x \frac{\partial}{\partial x_i} + \eta \frac{\partial}{\partial G}, \tag{6}$$

and solves the determining equations

$$\widehat{\mathrm{X}} \left(G_t + u_k G_{,k} - s_l \sqrt{G_{,n} G_{,n}} = 0 \right) \Big|_{G_t + u_k G_{,k} - s_l \sqrt{G_{,n} G_{,n}} = 0} = 0, \tag{7}$$

where $\widehat{\mathrm{X}}$ is the prolongation of X in (6) that includes components corresponding to derivatives [22, Chap. 1; 23, Chap. 2]. In (7) and below, we indicate derivatives of G by subindices. The time-derivative of G is denoted by $\frac{\partial G}{\partial t} \equiv G_t$, while the spatial derivatives are abbreviated by $\frac{\partial G}{\partial x_i} \equiv G_{,i}$, $\frac{\partial^2 G}{\partial x_i \partial x_j} \equiv G_{,ij}$, etc.

2.1 Point symmetries of the G-equation

First we note that the G-equation (1) involves the undefined vector function $\boldsymbol{u}$, the velocity of the underlying flow. Therefore symmetries of the G-equation essentially depend on the symmetry properties of equations defining $\boldsymbol{u}$. We now list point symmetries of the G-equation (1) for the two important cases; for details, see [18,19].

2.1.1 Point symmetries for arbitrary $\boldsymbol{u}$

One can show that for an arbitrary underlying flow velocity $\boldsymbol{u}$, in three space dimensions ($N = 3$), the G-equation (1) has point symmetries given by the following symmetry generators:

– translations in time and space

$$\mathrm{X}_1 = \frac{\partial}{\partial t}, \quad \mathrm{X}_2 = \frac{\partial}{\partial x_1}, \quad \mathrm{X}_3 = \frac{\partial}{\partial x_2}, \quad \mathrm{X}_4 = \frac{\partial}{\partial x_3}; \tag{8}$$

– scalings of independent variables

$$\mathrm{X}_5 = t\frac{\partial}{\partial t} + x_i\frac{\partial}{\partial x_i}; \tag{9}$$

– infinite symmetries

$$\mathrm{X}_R = K(G)\frac{\partial}{\partial G}, \tag{10}$$

where $K(G)$ is an arbitrary continuous function.

In particular, symmetries (10) in the global form (3) are written as

$$G^* = \mathcal{F}(G), \tag{11}$$

where $\mathcal{F}(G)$ is a largely arbitrary function subject to the restriction $\mathrm{d}\mathcal{F}(G)/\mathrm{d}G > 0$. As specific cases, the symmetries (10) include translations in G:

$$\mathrm{X}_T = \frac{\partial}{\partial G} \tag{12}$$

and scalings in G

$$\mathrm{X}_S = G\frac{\partial}{\partial G}. \tag{13}$$

Transformations (11) have important physical implications, and are employed within a novel modelling route in premixed turbulent combustion. In the literature, symmetry (11) is usually called *relabeling symmetry*.

In two space dimensions ($N = 2$), for an arbitrary propelling velocity $\boldsymbol{u}$, the G-equation (1) evidently has point symmetries $\mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_3, \mathrm{X}_5$ (8), (9) and X_R (10), which includes X_T (12) and X_S (13).

2.1.2 Point symmetries for $\boldsymbol{u} = const$

Due to a possible Galilean boost and hence without loss of generality, in this case, one may assume $\boldsymbol{u} = 0$. Under such restriction, the three-dimensional G-equation (1) has a wider class of point symmetries, as is detailed in [19]:

– Translations in time and space and scalings of independent variables, given by $\mathrm{X}_1, \ldots, \mathrm{X}_5$ (8), (9);
– three spatial rotations:

$$\mathrm{X}_6 = x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2}, \quad \mathrm{X}_7 = x_3\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_3}, \quad \mathrm{X}_8 = x_3\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_3}; \tag{14}$$

– three spacetime rotations (boosts):

$$\mathrm{X}_9 = x_1\frac{\partial}{\partial t} + s_l^2 t\frac{\partial}{\partial x_1},\quad \mathrm{X}_{10} = x_2\frac{\partial}{\partial t} + s_l^2 t\frac{\partial}{\partial x_2},\quad \mathrm{X}_{11} = x_3\frac{\partial}{\partial t} + s_l^2 t\frac{\partial}{\partial x_3}; \tag{15}$$

– four additional symmetries:

$$\begin{aligned}
\mathrm{X}_{12} &= tx_1\frac{\partial}{\partial t} + \frac{1}{2}\left(s_l^2t^2 + x_1^2 - x_2^2 - x_3^2\right)\frac{\partial}{\partial x_1} + x_1x_2\frac{\partial}{\partial x_2} + x_1x_3\frac{\partial}{\partial x_3},\\
\mathrm{X}_{13} &= tx_2\frac{\partial}{\partial t} + x_1x_2\frac{\partial}{\partial x_1} + \frac{1}{2}\left(s_l^2t^2 - x_1^2 + x_2^2 - x_3^2\right)\frac{\partial}{\partial x_2} + x_2x_3\frac{\partial}{\partial x_3},\\
\mathrm{X}_{14} &= tx_3\frac{\partial}{\partial t} + x_1x_3\frac{\partial}{\partial x_1} + x_2x_3\frac{\partial}{\partial x_2} + \frac{1}{2}\left(s_l^2t^2 - x_1^2 - x_2^2 + x_3^2\right)\frac{\partial}{\partial x_3},\\
\mathrm{X}_{15} &= \frac{1}{2}\left(s_l^2t^2 + x_1^2 + x_2^2 + x_3^2\right)\frac{\partial}{\partial t} + tx_1\frac{\partial}{\partial x_1} + tx_2\frac{\partial}{\partial x_2} + tx_3\frac{\partial}{\partial x_3};
\end{aligned} \tag{16}$$

– infinite relabelling symmetries X_R (10) ($K(G) \neq$ const);
– translations in G, given by X_T (12).

The reduction to two space dimensions is straightforward by limiting the admitted algebra to those symmetries independent of x_3, i.e., X_1, X_2, X_3, X_R, X_6, X_9, and X_{10}, and in addition X_{12}, X_{13}, and X_{15}, wherein $\partial/\partial x_3$ and x_3 is zero.

In addition to the above of point symmetries in [19] it has been proven that an infinite series of higher-order (Lie–Bäcklund) symmetries is admissible by (1) with $\boldsymbol{u} = 0$. Further details may be taken from the next subsection.

Note that Eq. 1 with $\boldsymbol{u} = 0$ has close links with other important equations known in mathematical physics such as the wave equation. Details are given in [19].

2.2 Higher-order symmetries of the G-equation

In mathematical physics it is known that many fundamental equations, such as the Burgers equation and the Korteweg–de Vries equation, admit wide classes of higher-order symmetries, sometimes called Lie–Bäcklund or Noether symmetries, in which, unlike Lie point symmetries, the infinitesimal components depend on derivatives of the dependent variables; e.g., [23, Chap. 5].

The actual derivation of the higher-order symmetries is almost identical to that of the classical point symmetries. However, the necessary algebra becomes increasingly more tedious for large orders of derivatives in the infinitesimals. For mathematical convenience, we here adopt the equivalent evolutionary form of the higher-order symmetries, where ξ^t and ξ^i may be set to zero if at least all first-order derivatives of G are included in η (see [23, Eq. 5.19]). Hence, without loss of generality, we seek the higher-order symmetries in the form

$$\tilde{\mathrm{X}} = \tilde{\eta}(t, \boldsymbol{x}, G, \partial G, \partial^2 G, \ldots)\frac{\partial}{\partial G}, \tag{17}$$

where $\partial^s G$ is a vector of all spatial derivatives of G of order s. In the present context, we may exclude all time derivatives of G since they can immediately be substituted using (1) and its differential consequences.

Keeping $\tilde{\eta}$ completely general and using the determining equations (7) with (17), we obtain

$$\widehat{\tilde{\mathrm{X}}}\left(G_t + u_k G_{,k} - s_l\sqrt{G_{,n}G_{,n}}\right)\Big|_{G_t + u_k G_{,k} - s_l\sqrt{G_{,n}G_{,n}}=0} = 0, \tag{18}$$

where

$$\widehat{\tilde{\mathrm{X}}} = \tilde{\eta}\frac{\partial}{\partial G} + (\mathrm{D}_t\tilde{\eta})\frac{\partial}{\partial G_t} + (\mathrm{D}_i\tilde{\eta})\frac{\partial}{\partial G_{,i}}$$

is the prolongation of $\tilde{\mathrm{X}}$ (17), and the total derivatives D_t and D_m are defined as

$$\mathrm{D}_t = \frac{\partial}{\partial t} + G_t\frac{\partial}{\partial G} + G_{t,i}\frac{\partial}{\partial G_{,i}} + G_{t,ij}\frac{\partial}{\partial G_{,ij}} + \cdots \tag{19}$$

and

$$\mathrm{D}_m = \frac{\partial}{\partial x_m} + G_{,m}\frac{\partial}{\partial G} + G_{,mi}\frac{\partial}{\partial G_{,i}} + G_{,mij}\frac{\partial}{\partial G_{,ij}} \cdots . \tag{20}$$

Note that in (18) we consider $\boldsymbol{u}$ as being a given function of space and time. Hence the dynamic equations determining $\boldsymbol{u}$ may not be considered in the present context.

After expanding (18) we may replace any term of the form $G_{t,ij\ldots}$ by the differential consequences of G_t taken from (1). This finally leads to a single determining equation for $\tilde{\eta}$ of the form

$$\frac{\partial\tilde{\eta}}{\partial t} + \left(u_i - s_l\frac{G_{,i}}{\sqrt{G_{,n}G_{,n}}}\right)\frac{\partial\tilde{\eta}}{\partial x_i} - \frac{\partial u_k}{\partial x_i}G_{,k}\frac{\partial\tilde{\eta}}{\partial G_{,i}} + \left[-\frac{\partial^2 u_k}{\partial x_i\partial x_j}G_{,k} - \frac{\partial u_k}{\partial x_i}G_{,kj} - \frac{\partial u_k}{\partial x_j}G_{,ki}\right.$$
$$\left. + s_l\left(\frac{G_{,im}G_{,jm}}{\sqrt{G_{,n}G_{,n}}} - \frac{G_{,im}G_{,m}G_{,jn}G_{,n}}{(G_{,k}G_{,k})^{3/2}}\right)\right]\frac{\partial\tilde{\eta}}{\partial G_{,ij}} + \cdots + \Delta_{i_1i_2\ldots i_p}(\partial G, \partial^2 G, \partial^p G)\frac{\partial\tilde{\eta}}{\partial G_{,i_1i_2\ldots i_p}} = 0, \tag{21}$$

where $\Delta_{i_1i_2\ldots i_p}$ comprises all the terms emerging from the differential consequences of (1). Several things are important to note about Eq. 21. No derivative of $\tilde{\eta}$ with respect to G appears, so any solution for $\tilde{\eta}$ can arbitrarily depend on G, since it is a free parameter of (21) in accordance with symmetry (10) or actually (11).

Most importantly, Eq. 21 is closed. We may readily verify this by choosing $\tilde{\eta}$ to depend only on derivatives of G up to the order p indicated by $\partial^p G$. Computing all differential consequences of (1), i.e., determining all $\Delta_{i_1i_2\ldots i_p}$ up to order p, we find that they contain derivatives of G only up to $\partial^p G$. Hence, Eq. 21 constitutes a linear hyperbolic equation in $\tilde{\eta}$ depending on the set of variables: $t, \boldsymbol{x}, G, G_{,i}, G_{,ij}, \ldots, G_{,i_1i_2\ldots i_p}$, where in fact G only appears as a parameter.

Solutions for $\tilde{\eta}$ with increasing derivative order $\partial^p G$ may be successively obtained, beginning with the lowest derivative order. First, we consider $\tilde{\eta}$ solely depending on G-derivatives up to order one. Hence, we limit $\tilde{\eta}$ to be a function of $t, \boldsymbol{x}, G$ and ∂G only. As a consequence (21) reduces to

$$\frac{\partial\tilde{\eta}}{\partial t} + \left(u_i - s_l\frac{G_{,i}}{\sqrt{G_{,n}G_{,n}}}\right)\frac{\partial\tilde{\eta}}{\partial x_i} - \frac{\partial u_k}{\partial x_i}G_{,k}\frac{\partial\tilde{\eta}}{\partial G_{,i}} = 0. \tag{22}$$

The characteristic system of equation (22) is

$$\frac{\mathrm{d}t}{\mathrm{d}\epsilon} = 1, \quad \frac{\mathrm{d}x_i}{\mathrm{d}\epsilon} = u_i - s_l\frac{G_{,i}}{\sqrt{G_{,n}G_{,n}}} \quad \text{and} \quad \frac{\mathrm{d}G_{,i}}{\mathrm{d}\epsilon} = -\frac{\partial u_k}{\partial x_i}G_{,k}. \tag{23}$$

which may be combined to

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = u_i - s_l\frac{G_{,i}}{\sqrt{G_{,n}G_{,n}}} \tag{24}$$

and

$$\frac{\mathrm{d}G_{,i}}{\mathrm{d}t} = -\frac{\partial u_k}{\partial x_i}G_{,k} = -\left(S_{ki} + W_{ki}\right)G_{,k}. \tag{25}$$

For purposes that will become clear later on we have decomposed the velocity gradient into a symmetric and a skew-symmetric part $\mathbf{S}$ and $\mathbf{W}$.

The latter two ordinary differential equations (ODE) of the characteristic system may by interpreted from a physical point of view. The first equation is simply the classical G-equation in explicit form, cf. (2), where the last term on the right-hand side is the normalized normal vector $n_i = -\frac{G_{,i}}{\sqrt{G_{,n}G_{,n}}}$ on an iso-G-surface. Hence we have

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = u_i + s_l n_i, \tag{26}$$

which is the original form due to [1]. It expresses the fact that the position vector $\boldsymbol{x}$ pointing to the surface is advected by the sum of the velocity field and the burning velocity propagating normal to the front.

Equation (25) states that the normal vector $G_{,i}$, not normalized here, changes its direction only by a non-zero velocity gradient. Since $G_{,i}$ is also a measure for the size of the surface element, we find that in a Lagrange-like coordinate system the propagation of the normal vector or alternatively the size of the surface element is independent of the burning velocity. This is evident from a physical point of view since propagation normal to itself does not increase the flame surface area.

Since for the solution of (26) along any characteristic only the normalized form of the normal vector is needed, we may rewrite (25) as

$$\frac{\mathrm{d}n_i}{\mathrm{d}t} = -\left(\delta_{il} - n_i n_l\right) \frac{\partial u_k}{\partial x_l} n_k. \tag{27}$$

The Eqs. 26 and 27 are fully equivalent to the system (24), (25). The right-hand side of (27) may be interpreted geometrically such that the velocity gradient on the front is projected into a tangential plane and the resulting vector projected into the front normal direction.

Although Eqs. 26, 27 or 24, 25 above are only defining the infinitesimal $\tilde{\eta}$, they have in fact a much broader foundation. For this we may take the fundamental definition of the normal vector in level-set form

$$\boldsymbol{n} = -\frac{\nabla G}{|\nabla G|} \tag{28}$$

and apply the time derivative on both sides. Second, we replace the terms on the right-hand side by differential consequences that emerge from the G-equation (1). Replacing all spatial gradients of G by (28), we finally obtain the evolution equation for the normal vector (here in Cartesian index notation)

$$\frac{\partial n_i}{\partial t} + \left(u_k + s_l n_k\right) \frac{\partial n_i}{\partial x_k} = -\left(\delta_{il} - n_i n_l\right) \frac{\partial u_k}{\partial x_l} n_k. \tag{29}$$

The latter equation possesses an identical right-hand side as (27). Redefining the left-hand side by the Lagrange-like operator

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \left(u_k + s_l n_k\right) \frac{\partial}{\partial x_k}, \tag{30}$$

we recover (27).

Note that the Lagrange-like formulation of the flame surface given by (26) and (27) constitutes a closed set of ODEs for a given $\boldsymbol{u}$. This is in contrast to (26) alone, which is a PDE. This may readily be observed by expressing $\boldsymbol{n}$ using surface-attached coordinates μ and λ (parametrization) and hence the normal vector is given by

$$\boldsymbol{n} = \frac{\frac{\partial \boldsymbol{x}}{\partial \lambda} \times \frac{\partial \boldsymbol{x}}{\partial \mu}}{\left| \frac{\partial \boldsymbol{x}}{\partial \lambda} \times \frac{\partial \boldsymbol{x}}{\partial \mu} \right|}. \tag{31}$$

Since (24) and (25) or alternatively (26) and (27) constitute a set of ODEs we conclude that infinitesimally small surface "elements" may be propagated independent of its adjacent surface elements. For a given velocity field $\boldsymbol{u}$ the system (24) and (25) may be readily integrated.

The system considerably simplifies for "homogeneous" flows, i.e., when the velocity is a linear function of $\boldsymbol{x}$ only:

$$u_i = A_{ij} x_j \quad \text{with} \quad A_{ij} = \frac{\partial u_i}{\partial x_j}, \tag{32}$$

where A_{ij} is at most a function of time. In this case (25) may be solved independently since it decouples from (24). Since the velocity gradient may always be uniquely decomposed into a symmetric and a skewsymmetric part $\mathbf{S}$ and $\mathbf{W}$, respectively, two extreme cases may be distinguished.

For $\mathbf{S} = 0$ we have pure rotation and hence there is no decrease or increase of surface area which corresponds to two imaginary eigenvalues of (25) with opposite sign and one null eigenvalue. Consequently, no flame surface area is generated or destructed.

The other case corresponds to a case with pure strain and no rotation $\mathbf{W} = 0$. Restricting to incompressible flows we find two real eigenvalues and a third one being the sum of the latter with opposite sign and hence leading to an increase or decrease of flame surface area.

It is very important to note at this point that from a geometrical interpretation of the characteristics, Eq. 1 may admit weak solutions. In this case two characteristics starting from the same initial iso-surface may cross at a later time. As a result a unique solution for the entire domain is no longer valid. The problems of weak or non-unique solutions are not discussed in this paper.

Extending the dependence of $\tilde{\eta}$ in the next step in (21) to the second derivatives $\partial^2 G$, we observe that (22) extends to

$$\frac{\partial\tilde{\eta}}{\partial t} + \left(u_i - s_l \frac{G_{,i}}{\sqrt{G_{,n}G_{,n}}}\right)\frac{\partial\tilde{\eta}}{\partial x_i} - \frac{\partial u_k}{\partial x_i}G_{,k}\frac{\partial\tilde{\eta}}{\partial G_{,i}} + \left[-\frac{\partial^2 u_k}{\partial x_i \partial x_j}G_{,k} - \frac{\partial u_k}{\partial x_i}G_{,kj} - \frac{\partial u_k}{\partial x_j}G_{,ki} + s_l\left(\frac{G_{,im}G_{,jm}}{\sqrt{G_{,n}G_{,n}}} - \frac{G_{,im}G_{,m}G_{,jn}G_{,n}}{(G_{,k}G_{,k})^{3/2}}\right)\right]\frac{\partial\tilde{\eta}}{\partial G_{,ij}} = 0. \tag{33}$$

The corresponding set of characteristic equations, Eq. 23, is expanded by

$$\frac{\mathrm{d}G_{,ij}}{\mathrm{d}t} = -\frac{\partial^2 u_k}{\partial x_i \partial x_j}G_{,k} - \frac{\partial u_k}{\partial x_i}G_{,kj} - \frac{\partial u_k}{\partial x_j}G_{,ki} + s_l\left(\frac{G_{,im}G_{,jm}}{\sqrt{G_{,n}G_{,n}}} - \frac{G_{,im}G_{,m}G_{,jn}G_{,n}}{(G_{,k}G_{,k})^{3/2}}\right), \tag{34}$$

where due to the first equation of (23) the dummy variable ϵ has already been replaced by t. In contrast to the equation for $G_{,i}$ or n_i, which does not contain the burning velocity, the latter equation for the spatial Hessian of G explicitly contains s_l.

For the purpose of analyzing (34) we employ the identity (see [19])

$$G^{-1}_{,ki}\frac{\mathrm{d}G_{,ij}}{\mathrm{d}t}G^{-1}_{,jl} = -\frac{\mathrm{d}G^{-1}_{,kl}}{\mathrm{d}t}, \tag{35}$$

where $G^{-1}_{,ij}$ is the matrix inverse of $G_{,ij}$ and $G_{,ik}G^{-1}_{,kj} = G^{-1}_{,ik}G_{,kj} = \delta_{ij}$. Multiplying (34) with $G^{-1}_{,ki}$ and $G^{-1}_{,jl}$ we find, using (35), that

$$\frac{\mathrm{d}G^{-1}_{,kl}}{\mathrm{d}t} = G^{-1}_{,ki}G^{-1}_{,jl}\frac{\partial^2 u_m}{\partial x_i x_j}G_{,m} + G^{-1}_{,ki}\frac{\partial u_l}{\partial x_i} + G^{-1}_{,jl}\frac{\partial u_k}{\partial x_j} - s_l\left(\frac{\delta_{kl}}{\sqrt{G_{,n}G_{,n}}} - \frac{G_{,k}G_{,l}}{(G_{,n}G_{,n})^{3/2}}\right). \tag{36}$$

This equation has the interesting property that a projection on $G_{,k}$ leads to an equation that is independent of s_l. To modify the resulting equation, to an evolution equation, we introduce the vector-valued quantity

$$H_i = G^{-1}_{,ki}G_{,k}, \tag{37}$$

characterizing the curvature effect of the surface. Note that $G^{-1}_{,ki}$ may be expressed in terms of $G_{,ki}$ itself using the matrix–tensor identity

$$G^{-1}_{,ij} = \frac{3}{2\lambda_3 - 3\lambda_2\lambda_1 + \lambda_1^3}\left[(\lambda_1^2 - \lambda_2)\delta_{ij} - 2\lambda_1 G_{,ij} + 2G^2_{,ij}\right], \tag{38}$$

with the eigenvalues $\lambda_1 = G_{,kk}$, $\lambda_2 = G^2_{,kk}$ and $\lambda_3 = G^3_{,kk}$ as is proved in [19].

Taking the temporal derivative of (37), using the product rule of differentiation and replacing $\frac{\mathrm{d}G_{,k}}{\mathrm{d}t}$ and $\frac{\mathrm{d}G^{-1}_{,kl}}{\mathrm{d}t}$ by its right-hand sides in (25) and (36) we obtain

$$\frac{\mathrm{d}H_l}{\mathrm{d}t} = G_{,k}G^{-1}_{,ki}G^{-1}_{,jl}\frac{\partial^2 u_m}{\partial x_i x_j}G_{,m} + \frac{\partial u_l}{\partial x_i}H_i. \tag{39}$$

For flows with weak curvature or homogeneous flows as defined in (32), the latter equation simplifies considerably since the first term on the right-hand side becomes negligible. Under this constraint (39) becomes a closed equation:

$$\frac{\mathrm{d}H_i}{\mathrm{d}t} = \frac{\partial u_i}{\partial x_k}H_k = (S_{ki} - W_{ki})\,H_k, \tag{40}$$

where the right-hand side has been rewritten employing $\mathbf{S}$ and $\mathbf{W}$. Although (40), appears to be quite similar to (25), there is one considerable difference, namely the sign of the strain rate $\mathbf{S}$. Physically this means that an increase of flame surface area due to $G_{,i}$ leads to a decrease of the curvature-related quantity H_i and vice versa.

Apart from the latter, any second-order geometric measure may be determined from the knowledge of $G_{,i}$ and $G_{,ij}$ such as the Gaussian or the mean curvature on the flame surface $G = G_0$. For instance, the mean curvature on a given point vx is given by

$$\kappa(t, \boldsymbol{x}) = \operatorname{div} \boldsymbol{n}$$

in terms of the normalized normal vector $\boldsymbol{n}$ defined in (28).

Note that, even in a Lagrange-like frame of reference that moves with the flame front, most of the curvature measures are not independent of s_l, unlike in the situation with a normal vector or H_i. For κ this is evident if one takes the divergence of (29). This leads to an s_l-dependent term which cannot be re-cast in the form of the Lagrange-like convection operator (30).

Remark 1 From the determining equations (22) and (33), one readily finds, respectively, first- and second-order symmetries of the G-equation (1) for the case $\boldsymbol{u} = 0$ as is pointed out in detail in [19]. Indeed, Eq. 22 reduces to the PDE

$$\frac{\partial \tilde{\eta}}{\partial t} - s_l \frac{G_{,i}}{\sqrt{G_{,n}G_{,n}}} \frac{\partial \tilde{\eta}}{\partial x_i} = 0,$$

and hence one obtains *an infinite set of first-order symmetries* given by

$$\tilde{\mathrm{X}}_{(1)} = \mathcal{G}\,(G, \partial G, \boldsymbol{C})\,\frac{\partial}{\partial G}, \quad \boldsymbol{C} = \boldsymbol{x} - s_l \boldsymbol{n} t, \tag{41}$$

where $\mathcal{G}\,(G, \partial G, \boldsymbol{C})$ is an arbitrary function.

The determining equations for the second-order symmetries reduce to the linear PDE

$$\frac{\partial \tilde{\eta}}{\partial t} - s_l \frac{G_{,i}}{\sqrt{G_{,n}G_{,n}}} \frac{\partial \tilde{\eta}}{\partial x_i} + \left[s_l \left(\frac{G_{,im}G_{,jm}}{\sqrt{G_{,n}G_{,n}}} - \frac{G_{,im}G_{,m}G_{,jn}G_{,n}}{(G_{,k}G_{,k})^{3/2}} \right) \right] \frac{\partial \tilde{\eta}}{\partial G_{,ij}} = 0,$$

which yields *an infinite set of second-order symmetries* given by

$$\tilde{\mathrm{X}}_{(2)} = \mathcal{H}(G, \partial G, \boldsymbol{C}, \mathbf{D}) \frac{\partial}{\partial G}, \tag{42}$$

where $\mathcal{H}(G, \partial G, \boldsymbol{C}, \mathbf{D})$ is an arbitrary function, and the tensor $\mathbf{D}$ is given by

$$D_{ij} = s_l \left[\frac{\delta_{ij}}{\sqrt{G_{,m}G_{,m}}} - \frac{G_{,i}G_{,j}}{(G_{,m}G_{,m})^{3/2}} \right] t + \frac{3}{2\lambda_3 - 3\lambda_2\lambda_1 + \lambda_1^3} \left[(\lambda_1^2 - \lambda_2)\delta_{ij} - 2\lambda_1 G_{,ij} + 2G_{,ik}G_{,kj} \right], \tag{43}$$

with λ_i given below Eq. 38.

Although not obvious, one may show that that all point symmetries (8)–(10), (14)–(16) of the G-equation (1) with $\boldsymbol{u} = 0$ are contained in the first- and second-order symmetries (41), (42). For example, the point symmetry X_9 (15) is found from (41) with $\mathcal{G}\,(G, \partial G, \boldsymbol{C}) = s_l \sqrt{G_{,n}G_{,n}} C_1$.

3 Conservation laws of the G-equation

Within the current section, we consider the conservation properties of the G-equation (1). Recall that the derivation of the G-equation is exclusively based on the kinematic balance equation (2). The latter equation has no link to any classical conservation law and to the best of the authors' knowledge no conservation laws or related Casimirs are known for the G-equation.

We seek local conservation laws of the G-equation (1) of the form

$$\mathrm{D}_t \Psi[G, \boldsymbol{u}] + \mathrm{D}_k \Phi_k[G, \boldsymbol{u}] = 0, \tag{44}$$

where D_t and D_k are defined by (19), (20). The density $\Psi[G, \boldsymbol{u}]$ and the fluxes $\Phi_k[G, \boldsymbol{u}]$ can depend on $t, \boldsymbol{x}, G$, the underlying flow velocity $\boldsymbol{u}$, and partial derivatives of G and $\boldsymbol{u}$. (The dependence on G and/or its derivatives is essential.)

In particular, we are interested in finding local conservation laws of the G-equation (1) in the following settings, both in two and three spatial dimensions:

- local conservation laws of (1) holding for arbitrary propelling velocities $\boldsymbol{u}(t, \boldsymbol{x})$;
- local conservation laws of (1) coupled to fluid-dynamics equations satisfied by $\boldsymbol{u}$, e.g., incompressible Euler or Navier–Stokes equations;
- local conservation laws of (1) for $\boldsymbol{u} = 0$ (equivalently, $\boldsymbol{u} = \text{const}$);
- local conservation laws of (1) for $s_l \neq \text{const}$.

For the derivation of local conservation laws, we employ the *direct method* [20–22]. The direct method consists in finding local *multipliers* of the PDEs of the given system, depending on independent and dependent variables of the given system and derivatives of dependent variables, such that the corresponding linear combination yields a divergence expression (44).

In particular, for PDE systems which can be written in a solved form with respect to some leading derivatives (which is the case for the majority of physical PDE systems), it is well-known that *all* its local conservation laws (44) arise from local multipliers, provided that a sufficiently general ansatz for multipliers has been chosen.

To generate determining equations for local conservation-law multipliers, the direct method employs Euler differential operators. In this paper, determining equations for multipliers were written and solved using the automated symbolic software package GeM for Maple [24].

First, consider the G-equation (1) coupled to incompressible Euler ($\nu = 0$) or Navier–Stokes ($\nu \neq 0$) equations, in two dimensions:

$$\begin{aligned}
&G_t + u_k G_{,k} = s_l \sqrt{G_{,k} G_{,k}}, \\
&(u_i)_t + u_k (u_i)_{,k} = -\frac{1}{\rho} p_{,i} + \nu \Delta u_i, \quad i = 1, 2, \\
&(u_1)_{,1} + (u_2)_{,2} = 0,
\end{aligned} \tag{45}$$

In (45), $p(t, \boldsymbol{x})$ is the fluid pressure, and $\rho = \text{const}$ is the fluid density. To find local conservation laws of the system (45) using the direct method, one seeks quadruples of multipliers $\Lambda_q[\hat{G}, \hat{\boldsymbol{u}}, \hat{p}] = \Lambda_q(t, \boldsymbol{x}, \hat{G}, \hat{\boldsymbol{u}}, \hat{p}, \partial \hat{G}, \partial \hat{\boldsymbol{u}}, \partial \hat{p}, \ldots)$, $q = 1, 2, 3, 4$, such that the linear combination yields a divergence expression,

$$\begin{aligned}
&\Lambda_1[\hat{G}, \hat{\boldsymbol{u}}, \hat{p}] \left(\hat{G}_t + \hat{u}_k \hat{G}_{,k} - s_l \sqrt{\hat{G}_{,k} \hat{G}_{,k}} \right) + \Lambda_2[\hat{G}, \hat{\boldsymbol{u}}, \hat{p}] \left((\hat{u}_1)_t + \hat{u}_k (\hat{u}_1)_{,k} + \frac{1}{\rho} \hat{p}_{,1} - \nu \Delta \hat{u}_1 \right) \\
&\quad + \Lambda_3[\hat{G}, \hat{\boldsymbol{u}}, \hat{p}] \left((\hat{u}_2)_t + \hat{u}_k (\hat{u}_2)_{,k} + \frac{1}{\rho} \hat{p}_{,2} - \nu \Delta \hat{u}_2 \right) + \Lambda_4[\hat{G}, \hat{\boldsymbol{u}}, \hat{p}] \left((\hat{u}_1)_{,1} + (\hat{u}_2)_{,2} \right) \\
&\quad \equiv \mathrm{D}_t \Psi[\hat{G}, \hat{\boldsymbol{u}}, \hat{p}] + \mathrm{D}_k \Phi_k[\hat{G}, \hat{\boldsymbol{u}}, \hat{p}],
\end{aligned} \tag{46}$$

for arbitrary functions $\hat{G}, \hat{\boldsymbol{u}}, \hat{p}$. Then for solutions ($\hat{G} = G(t, \boldsymbol{x}), \hat{\boldsymbol{u}} = \boldsymbol{u}(t, \boldsymbol{x}), \hat{p} = p(t, \boldsymbol{x})$) of the PDE system (45), multipliers $\Lambda_q[G, \boldsymbol{u}, p]$, $q = 1, 2, 3, 4$, yield a conservation law (44).

However, the following negative result is established by direct computation.

Theorem 1 *The PDE system* (45) *in two spatial dimensions has no G-dependent local conservation laws arising from multipliers depending on $t, x_1, x_2, u_1, u_2, p, G, p_t, \partial G, \partial u_1, \partial u_2, \partial p, \partial^2 G$ and $\partial^2 P$.*

Corollary 1 *The G-equation* (1) *in two spatial dimensions does not have conservation laws holding for* arbitrary *propelling velocities $\boldsymbol{u}(t, \boldsymbol{x})$, arising from multipliers depending on $t, x_1, x_2, G, \partial G$ and $\partial^2 G$.*

We note that multipliers of all usual (G-independent) local conservation laws of fluid dynamics (i.e., the conservation of mass for $\nu \neq 0$, and the conservation of mass, momentum, energy, and angular momentum for $\nu \neq 0$) were recovered in the computation for Theorem 1.

Remark 2 Corollary 1 also holds for the G-equation (1) in three spatial dimensions.

However, one can show that an infinite number of conservation laws arises for the G-equation (1) with $\boldsymbol{u} = \text{const}$, both in two and three dimensions. Here one seeks a multiplier $\Lambda[G]$ such that

$$\Lambda[\hat{G}]\left(\hat{G}_t + \hat{u}_k \hat{G}_{,k} - s_l\sqrt{\hat{G}_{,k}\hat{G}_{,k}}\right) \equiv \mathrm{D}_t \Psi[\hat{G}] + \mathrm{D}_k \Phi_k[\hat{G}] \tag{47}$$

holds for an arbitrary function $\hat{G}(t, \boldsymbol{x})$. The following theorem holds.

Theorem 2 *The G-equation (1) with $\boldsymbol{u} = 0$ ($\boldsymbol{u} = \text{const}$), in two and three spatial dimensions, has an infinite number of second-order conservation laws, i.e., conservation laws corresponding to multipliers of the form $\Lambda[G] = \Lambda(t, \boldsymbol{x}, G, \partial G, \partial^2 G)$.*

It can be shown that no conservation laws arise for the G-equation from purely first-order multipliers $\Lambda[G] = \Lambda(t, \boldsymbol{x}, G, \partial G)$.

In Sect. 3.1 below, we completely classify second-order local conservation laws of the G-equation arising from multipliers $\Lambda[G] = \Lambda(G, \partial G, \partial^2 G)$ in two dimensions. In three dimensions (Sect. 3.2), such a classification presents a significant computational difficulty, therefore we limit ourselves to finding infinite families of conservation laws generalizing the families found for the two-dimensional case.

Remark 3 It easy to show that the G-equation (1) with $s_l = s_l(t)$ also has an infinite number of local conservation laws, since one may redefine the time variable according to $\mathrm{d}\tau = s_l(t)\mathrm{d}t$, and convert (1) to an equivalent one with $s_l = 1$. The more general case $s_l = s_l(t, \boldsymbol{x})$ requires conservation law classification and no general conserved quantity exists for arbitrary $s_l(t, \boldsymbol{x})$.

Remark 4 Although by Theorem 1, within the multiplier ansatz specified therein, there are no local conservation laws of the G-equation (1) for a general flow velocity $\boldsymbol{u}$ satisfying incompressible Euler or Navier–Stokes equations, one may easily find *particular* velocity distributions (which can satisfy Euler or Navier–Stokes equations), for which even infinite numbers of conservation laws exist! For example, this is the case for $\boldsymbol{u} = (0, x)$. However, a classification of conservation laws of the G-equation (1) with respect to $\boldsymbol{u}(t, \boldsymbol{x})$ as a constitutive function is a highly computationally intensive task, and is out of scope of this paper.

3.1 Local conservation laws of the G-equation (1) with $\boldsymbol{u} = 0$ in the two-dimensional case

In two dimensions, with no underlying flow velocity, the G-equation (1) takes the form

$$G_t = s_l\sqrt{G_{,1}^2 + G_{,2}^2}. \tag{48}$$

We look for conservation laws arising from second order local conservation law multipliers

$$\Lambda[G] = \Lambda(G, G_{,1}, G_{,2}, G_{,11}, G_{,12}, G_{,22}). \tag{49}$$

The solution of the corresponding determining equations (omitted because of extensive length, see [22]), yields the following result.

Theorem 3 *Consider the two-dimensional G-equation* (48)*, and multipliers of the form* (49)*. The complete set of such multipliers yielding local conservation laws of the G-equation* (48) *is exhausted by the following classes.*

$$\Lambda^{(1)}[G] = \frac{1}{G_{,2}^3}(G_{,2}G_{,12} - G_{,1}G_{,22}); \tag{50}$$

$$\Lambda^{(2)}[G] = \frac{1}{G_{,1}^3}(G_{,2}G_{11} - G_{,1}G_{12}); \tag{51}$$

$$\Lambda^{(3)}[G] = F(G_{,1}, G_{,2})H_2; \tag{52}$$

$$\Lambda^{(4)}[G] = \frac{1}{G_{,1}^3} R\left(\frac{G_{,2}}{G_{,1}}\right)\left(G_{,2}^2 G_{,11} + G_{,1}^2 G_{,22} - 2G_{,1}G_{,2}G_{,12}\right); \tag{53}$$

$$\begin{aligned}\Lambda^{(5)}[G] = \frac{1}{G_{,1}^2}\Bigg(&\frac{G_{,1}}{G_{,2}}\left[\frac{G_{,1}}{G_{,2}} Q\left(\frac{G_{,2}}{G_{,1}}\right) - Q'\left(\frac{G_{,2}}{G_{,1}}\right)\right] G_{,22} \\ &+ 2Q'\left(\frac{G_{,2}}{G_{,1}}\right) G_{,12} - \left[Q\left(\frac{G_{,2}}{G_{,1}}\right) + \frac{G_{,2}}{G_{,1}} Q'\left(\frac{G_{,2}}{G_{,1}}\right)\right] G_{,11}\Bigg),\end{aligned} \tag{54}$$

where F, R and Q are arbitrary sufficiently smooth functions of their arguments, and

$$H_2 = \begin{vmatrix} G_{,11} & G_{,12} \\ G_{,12} & G_{,22} \end{vmatrix} \tag{55}$$

is the two-dimensional spatial Hessian determinant of G.

One can check that the set of multipliers (50)–(54) is closed under the interchange of the spatial variables $x_1 \leftrightarrow x_2$.

3.1.1 Computation of densities and fluxes

We now find explicit forms of conservation laws using the multipliers (50)–(54).

Since the G-equation (48) has a scaling symmetry (13), it is scaling-invariant. A method of computation of density and fluxes of scaling-invariant conservation laws of scaling-invariant PDE systems was proposed in [25] (see also [22,26]). Within this method, formulas for density and fluxes are rather simple and involve only differentiation.

Here we list density/flux formulas only for first-order PDEs. Let $\Lambda[G]$ be a local conservation law multiplier, and X (6) a local symmetry of the PDE (48). In evolutionary form (17), a local symmetry generator (6) is written as

$$\tilde{\mathrm{X}} = \tilde{\eta}[G]\frac{\partial}{\partial G}, \quad \tilde{\eta}[G] = \eta^G - G_t\xi^t - G_{,i}\xi_i^x. \tag{56}$$

In particular, if the symmetry (56) is a scaling symmetry, it has the form

$$\tilde{\mathrm{X}}_{scal} = \tilde{\eta}[G]\frac{\partial}{\partial G} \equiv (qG - G_t pt - G_{,i}p^i x_i)\frac{\partial}{\partial G}, \qquad q,\, p,\, p^i = \text{const}. \tag{57}$$

Then the density and the fluxes of the local conservation law (44) corresponding to the multiplier $\Lambda[G]$ (cf. (47)) are given (up to conservation law equivalence) by

$$\Psi[G] = \tilde{\eta}[G]\Lambda[G]\frac{\partial R[G]}{\partial G_t}, \quad \Phi^i[G] = \tilde{\eta}[G]\Lambda[G]\frac{\partial R[G]}{\partial G_{,i}}, \quad i = 1, 2, \tag{58}$$

where $R[G] = s_l^{-1}G_t - \sqrt{G_{,1}^2 + G_{,2}^2}$.

Remark 5 The only restrictions in applying formulas (58) are that the conservation law being sought must be *scaling-invariant* and *noncritical with respect to* X_S, i.e.,

$$\chi = \lambda + r + p + \sum_{i=1}^{n} p^i \neq 0, \tag{59}$$

where λ and r are constant scaling weights of the multiplier and the G-equation:

$$\mathrm{X}_{scal}\Lambda[G] = \lambda\Lambda[G], \quad \mathrm{X}_{scal}R[G] = rR[G].$$

If a conservation law is critical with respect to X_S, i.e., $\chi = 0$, then formulas (58) yield density and fluxes of a *trivial* (and thus useless) conservation law. (For details, see [22,25,26].)

Remark 6 Formulas (58) can be used with an *arbitrary* local symmetry (56) of a given PDE, i.e., not necessarily a scaling symmetry. In that case, formulas (58) yield density and fluxes of a conservation law which is generally *different* from the one corresponding to the multiplier $\Lambda[G]$ (cf. (47)). For further details, see [22,25,26].

Also note that in general, the symmetry involved in the formula (58) does not have to be a point symmetry, but may be a contact or a higher-order symmetry as is explicitly given for (48) or (63) up to order two in (42).

Since $\partial R[G]/\partial G_{,i} = n_i$, where $\boldsymbol{n} = (n_1, n_2)$ is the unit normal (28) to the flame front, each conservation law found through the formula (58) has a general form

$$\mathrm{D}_t(s_l^{-1}\tilde{\eta}[G]\Lambda[G]) + \mathrm{D}_k\,(n_k\tilde{\eta}[G]\Lambda[G]) = 0. \tag{60}$$

The two-dimensional G-equation (48) has the following point symmetries:

- translations in G given by X_T (12);
- Translations and scalings $\mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_3, \mathrm{X}_5$ given by (8)–(9);
- relabelling symmetry X_R given by (10);
- rotations X_6 and boosts $\mathrm{X}_9, \mathrm{X}_{10}$ given by (14), (15);
- 2D version of the symmetries $\mathrm{X}_{12}, \mathrm{X}_{13}$ and X_{15} given by (16) ($\partial/\partial x_3 = 0, x_3 = 0$).

These symmetries, together with multipliers $\Lambda^{(i)}[G]$ ($i = 1, \ldots, 5$) given by (50)–(54), yield the total of $12 \times 5 = 60$ classes of conservation laws of the two-dimensional G-equation (48). This set, however, contains some trivialities and redundancies, as explained below:

1. for each of the multipliers $\Lambda^{(1)}[G]$ (50), $\Lambda^{(2)}[G]$ (51), one has 11 scalar conservation laws corresponding to single point symmetries, and an infinite family of conservation laws corresponding to the family of relabelling symmetries X_R (10);
2. from the multipliers $\Lambda^{(3)}[G]$ (52), one has 10 infinite families of conservation laws, since conservation laws arising from translation symmetries $\mathrm{X}_2, \mathrm{X}_3$ (8) also arise from symmetry X_T (12) for a different form of $F(G_{,1}, G_{,2})$;
3. from the multipliers $\Lambda^{(4)}[G]$ (53) and $\Lambda^{(5)}[G]$ (54), one obtains 12 infinite families of conservation laws corresponding to the above 12 point symmetries;
4. for each multiplier $\Lambda^{(i)}[G]$, scaling symmetries X_5 (9) and X_S (13) (i.e., X_R with $K(G) = G$) yield equivalent conservation laws;
5. for multipliers $\Lambda^{(1,2)}[G]$ and the scaling symmetry X_S (13), one has $p = p^i = 0$. For this symmetry, one readily finds the scaling weight $\lambda = -1$ for each of the multipliers $\Lambda^{(1,2)}[G]$. The corresponding scaling weight of the G-equation here is $r = 1$. Therefore one has $\chi = -1 + 1 = 0$, and it follows that the corresponding conservation law is critical. Its density and fluxes are thus not obtainable from formulas (58), i.e., the conservation law given by the expression (60) is trivial;
6. there might be further redundancies and trivial conservation laws within the above indicated conservation law set. These will be studied in detail in future work.

Remark 7 In the formula (60), one can indeed use higher-order symmetries $\tilde{\eta}[G]$. Using the known infinite family of second-order symmetries (42) (which includes first-order and point symmetries) of G-equation (48), we obtain a wider class of conservation laws

$$\mathrm{D}_t\left(s_l^{-1}\Lambda^{(i)}[G]\mathcal{H}(G, \partial G, \boldsymbol{C}, \mathbf{D})\right) + \mathrm{D}_k\left(n_k\mathcal{H}(G, \partial G, \boldsymbol{C}, \mathbf{D})\Lambda^{(i)}[G]\right) = 0, \quad i = 1, \ldots, 5, \tag{61}$$

satisfied by all solutions of the G-equation (48). Here $\Lambda^{(i)}[G]$ are given by (50)–(54). From (61) we may derive conserved densities know as classical measures from differential geometry. For instance, we may choose R in (53) and $\mathcal{H}$ in (61) such that we obtain

$$\Lambda^{(4)}[G]\mathcal{H}(G, \partial G, \boldsymbol{C}, \mathbf{D}) \equiv \kappa(G) = -\frac{\boldsymbol{t}(G) \cdot \mathbf{H}(G) \cdot \boldsymbol{t}(G)^T}{|\nabla G|}, \tag{62}$$

where κ is the curvature of a plane implicit curve and $\boldsymbol{t}(G)$ and $\mathbf{H}(G)$ are, respectively, the normalized tangent vector and the Hessian matrix of G:

$$\boldsymbol{t}(G) = \frac{\nabla G}{|\nabla G|}, \quad \mathbf{H}(G) = \begin{pmatrix} G_{xx} & G_{xy} \\ G_{xy} & G_{yy} \end{pmatrix}.$$

Even vector-valued quantities such as $\mathbf{H}(G) \cdot \boldsymbol{t}(G)^T$, normalized adequately, may be derived from (50) and (51) as conserved densities.

3.2 Local conservation laws of the G-equation (1) with $\boldsymbol{u} = 0$ in the three-dimensional case

In three dimensions with no flow, the G-equation (1) takes the form

$$G_t = s_l \sqrt{G_{,1}^2 + G_{,2}^2 + G_{,3}^2}. \tag{63}$$

We look for conservation laws arising from second-order conservation law multipliers

$$\mathrm{M}[G] = \mathrm{M}(G, G_{,1}, G_{,2}, G_{,3}, G_{,11}, G_{,12}, G_{,13}, G_{,22}, G_{,23}, G_{,33}). \tag{64}$$

The following theorem holds:

Theorem 4 *The three-dimensional G-equation (63) has local conservation laws arising from the following multipliers of the form (64):*

1. *the multiplier*

$$\mathrm{M}^{(1)}[G] = \frac{G_{,1}G_{,2}G_{,3}}{(G_{,2}^2 + G_{,3}^2)^2}(G_{,22} - G_{,33}) - \frac{G_{,1}(G_{,2}^2 - G_{,3}^2)}{(G_{,2}^2 + G_{,3}^2)^2} G_{,23} + \frac{G_{,2}G_{,13} - G_{,3}G_{,12}}{G_{,2}^2 + G_{,3}^2} \tag{65}$$

   and two additional multipliers $\mathrm{M}^{(2)}[G]$, $\mathrm{M}^{(3)}[G]$ obtained by the two cyclic permutations $x_1 \to x_2 \to x_3 \to x_1$;
2. *the multiplier*

$$\mathrm{M}^{(4)}[G] = \frac{G_{,3}G_{,11} - G_{,1}G_{,13}}{2(G_{,1}^2 + G_{,3}^2)} + \frac{G_{,2}G_{,23} - G_{,3}G_{,22}}{2(G_{,2}^2 + G_{,3}^2)} + \frac{G_{,3}(G_{,1}^2 - G_{,2}^2)G_{,33}}{2(G_{,1}^2 + G_{,3}^2)(G_{,2}^2 + G_{,3}^2)} \tag{66}$$

   and two additional multipliers $\mathrm{M}^{(5)}[G]$, $\mathrm{M}^{(6)}[G]$ obtained by the two cyclic permutations $x_1 \to x_2 \to x_3 \to x_1$;
3. *the family of multipliers*

$$\mathrm{M}^{(7)}[G] = F(G_{,1}, G_{,2}, G_{,3})H_3, \tag{67}$$

   where $F(G_{,1}, G_{,2}, G_{,3})$ is an arbitrary sufficiently smooth function of its arguments, and

$$H_3 = \begin{vmatrix} G_{,11} & G_{,12} & G_{,13} \\ G_{,12} & G_{,22} & G_{,23} \\ G_{,13} & G_{,23} & G_{,33} \end{vmatrix} \tag{68}$$

   is the three-dimensional Hessian determinant of G.

The complete classification of local conservation-law multipliers of the form (64) has not been done in this paper because of the computational complexity.

To construct the actual conservation laws, we use the same technique as for the 2D case. For each pair of local symmetry in the evolutionary form (56) and a local conservation law multiplier $\mathrm{M}[G]$, one obtains a conservation law (60) (with summation in k now taken from 1 to 3.)

Equation (63) has the point symmetries $\mathrm{X}_1, \ldots, \mathrm{X}_{15}$ given by (8), (9), (14)–(16), relabelling symmetries X_R (10) ($K(G) \neq$ const), and translations in G given by X_T (12).

These symmetries, together with multipliers $\mathrm{M}^{(i)}[G]$ ($i = 1, \ldots, 7$) (65)–(67), yield a total of $17 \times 7 = 119$ classes of conservation laws of the two-dimensional G-equation (48). In particular, one obtains an infinite number of conservation arising from infinite relabelling symmetries X_R and/or multipliers from an infinite family (67).

Again, some of these conservation laws are equivalent. Similarly to the 2D case, for each $\mathrm{M}^{(i)}[G]$, equivalent conservation laws arise from scaling symmetries X_5 (9) and X_S (13) (i.e., X_R with $K(G) = G$). Moreover, from the multipliers $\mathrm{M}^{(7)}[G]$ (67), conservation laws arising from translation symmetries $\mathrm{X}_2, \mathrm{X}_3$ (8) also arise from symmetry X_T (12) for a different form of $F(G_{,1}, G_{,2}, G_{,3})$.

Remark 8 Similarly to the two-dimensional case, one obtains a much wider class of conservation laws for the three-dimensional G-equation (63) by considering higher-order symmetries $\tilde{\eta}[G]$, in particular, the infinite family of second-order symmetries (42). It follows that the G-equation (63) has seven sets of conservation laws given by

$$\mathrm{D}_t\left(\mathrm{M}^{(i)}[G]\mathcal{H}(G, \partial G, \boldsymbol{C}, \mathbf{D})\right) + \mathrm{D}_k\left(n_k\mathcal{H}(G, \partial G, \boldsymbol{C}, \mathbf{D})\mathrm{M}^{(i)}[G]\right) = 0, \quad i = 1, \ldots, 7, \tag{69}$$

where $\mathrm{M}^{(i)}[G]$ are given by (65)–(67).

In summary, an infinite system of conservation laws has been generated, both for the two- and three-dimensional G-equation in the limit of $s_l = \text{const}$ (equivalently, $s_l = s_l(t)$) and $\boldsymbol{u} = 0$ (equivalently, $\boldsymbol{u} = \text{const}$).

Both symmetry and conservation-law properties of the previous and the present section are employed in the subsequent section for the construction of new numerical schemes solving the G-equation.

4 Application to the construction of numerical schemes

In the introduction it has been briefly pointed out that a broad variety of numerical schemes have been derived for the G-equation and level-set equations in general. The level-set method was devised by Osher and Sethian [7] as a simple and versatile method for computing and analyzing the motion of an interface in two or three dimensions. In short, the key advantage of the level set method is due to the fact that, in general, it allows for advancing of interfaces that change topology, formate corners, cusps, and singularities, as well as it elegantly models the interplay between different disciplines such as flow physics and chemistry on the interface.

A detailed review on the different numerical methods for solving level-set equations is given in [27] and, more specifically, for fluid interfaces in [28].

Essentially all schemes rely on two global steps during the time advancement. In the first step the level-set equation, here the G-equation, is advanced in time. Although, in principle, advancement is only needed for the front level-set $G = G_0$, practically it has to be advanced either in a narrow band close to the front or even in the entire domain of integration. Depending on the geometrical complexity of the level-set, the advancement of the neighbouring level-set functions in tandem with the one corresponding to the zero level-set—the extension velocity, and the actual model equations a variety of numerical difficulties may arise. These can include issues such as the development of singularities which are non-differentiable along the level-set or small slopes normal to the level-set which decreases the numerical accuracy for capturing the position of the level-set. Because of the latter and related issues, the actual computation may completely break down after a certain number of time steps.

For this reason, a second step usually has to be invoked, i.e., after one or several time steps, a reinitialization of the neighboring level-set functions is executed, such that the gradient normal to the zero level-set is set to equal one.

In other words, the signed distance function equation $|\nabla G| = 1$ is solved. This leads to a smooth unique G-field sufficiently regular for the next time step. These and related techniques solve the problem emerging from the first step though implying new difficulties, as pointed out in [28]. First, reinitialization tends to generate some error in the position of the front, and these errors can lead to inaccuracy. As a result, the general problem of mass loss during long calculations may become worse. Second, poor programming of reinitialization schemes, adaptive strategies, and extension schemes can easily render an efficient algorithm inefficient. In conclusion, reinitialization may only be adapted if necessary and as seldom as possible.

Apart from the problems pointed out in the introduction such as stability or high computational costs, most of the classical schemes have an additional disadvantage that the computation of curvature terms becomes rather inaccurate. This is primarily due to the fact that in most applications, second-order finite-difference or finite- volume schemes are employed. Hence the computation of curvature terms which are second order in derivative may become rather inaccurate. This in fact may lead to spurious results.

We argue that both of the proposed methods are completely different from what has been developed so far and may account for some of these problems. There are two key differences. First, we do not only advance a single scalar level-set equation but instead may propagate higher-order geometrical terms such as the normal vector or the spatial Hessian of G. Hence, it is natural to deduce that by invoking these quantities higher-order numerical accuracies may be achieved.

Second, for the schemes to be derived in Sect. 4.1, we note that only conservation laws need to be solved. For this reason, we may hark back to almost any scheme that has been developed in the last decades for nonlinear conservation laws. Many of these schemes can cope with singularies if, e.g., flux or slope limiters are invoked.

Still, one cannot conceal that certain new problems may arise. First of all, it is quite clear, as it has been pointed out above, that any derivative to be computed needs to be continuous. This may either limit the possible level-set geometries or, if employed in the context of a splitting scheme, one part of the equation will need to assure smoothness up to the needed derivative order (details are discussed below). A second problem may arise from gradients normal to the zero level-set. Though many robust and flexible schemes for nonlinear conservation laws may cope with singularities, i.e., high gradients in the numerical scheme, the computation may still need some reinitialization after certain time steps. This may not be judged on without any actual testing.

4.1 Numerical schemes based on conservation laws

Families of conservation laws for the two-dimensional and three-dimensional G-equations (48) and (63) for $\boldsymbol{u} = 0$ obtained in Sects. 3.1 and 3.2 can be directly used for the construction of numerical schemes, as described below. Suppose at $t = 0$, initial data for the function G and its first and second spatial partial derivatives is provided. Values on the next time step are denoted by tildas.

4.1.1 Two dimensions

In two dimensions, one can use different families of local conservation laws (60) to obtain values for the function $G(t, \boldsymbol{x})$ and its first and second spatial partial derivatives on the next time step. For example, one can proceed as follows:

1. the conservation law arising from the symmetry X_T (12) and the multiplier $\Lambda^{(3)}[G]$ (52) with $F(G_{,1}, G_{,2}) = 1$ yields the Hessian determinant $\widetilde{H_2}$ (55);
2. the conservation law arising from the symmetry X_R (10) with $K(G) = G$ and the multiplier $\Lambda^{(3)}[G]$ (52) with $F(G_{,1}, G_{,2}) = 1$ yields $\widetilde{G}\widetilde{H_2}$, and hence, employing $\widetilde{H_2}$ from step 1, one obtains $\widetilde{G}$;
3. the conservation law arising from the symmetry X_T (12) and the multiplier $\Lambda^{(3)}[G]$ (52) with $F(G_{,1}, G_{,2}) = G_{,1}$ and $G_{,2}$, respectively, yield $\widetilde{G_{,1}}\widetilde{H_2}$ and $\widetilde{G_{,2}}\widetilde{H_2}$, and hence, again employing $\widetilde{H_2}$ from step 1, $\widetilde{G_{,1}}$ and $\widetilde{G_{,2}}$;

4. let $A = G_{,1}G_{,22} - G_{,2}G_{,12}$, $B = G_{,2}G_{,11} - G_{,1}G_{,12}$. Then the conservation laws arising from the symmetry X_T (12) and the multipliers $\Lambda^{(1)}[G]$ (50), $\Lambda^{(2)}[G]$ (51) yield quantities $\widetilde{A}$ and $\widetilde{B}$ at the next time step. Quantities $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{H}$ can be solved to yield the second partial derivatives $\widetilde{G_{,11}}$, $\widetilde{G_{,12}}$ and $\widetilde{G_{,22}}$ on the next time step.

If only G and its first spatial derivatives are required, one can limit the algorithm to the first three steps, which considerably reduces the number of computations. In this case the computation of the second-order spatial derivative terms are not needed at all.

In this case it is also possible to modify the algorithm as such that very small values for $\sqrt{G_{,1}^2 + G_{,2}^2}$ are avoided which may lead to computational difficulties in the spatial flux terms. This may easily be achieved by introducing $F(G_{,1}, G_{,2}) \to F(G_{,1}, G_{,2})\sqrt{G_{,1}^2 + G_{,2}^2}$ which avoids division by near zero values.

4.1.2 Three dimensions

In three dimensions, values for the function $G(t, \boldsymbol{x})$ and its first and second spatial partial derivatives are obtained in a similar fashion. Here one can use the larger set of available conservation laws (60) in three dimensions, to compute the three first and the six second spatial partial derivatives of $G(t, \boldsymbol{x})$ on the next time step. For example, one can proceed as follows:

1. use the conservation law arising from the translational symmetry X_T (12) $\mathrm{M}^{(7)}[G]$ (67) with $F(G_{,1}, G_{,2}, G_{,3}) = 1$ to obtain the Hessian determinant $\widetilde{H_3}$ (68);
2. use the conservation law the conservation law arising from the relabelling symmetry X_R (10) with $K(G) = G$ (i.e., scaling symmetry) and $\mathrm{M}^{(7)}[G]$ (67) with $F(G_{,1}, G_{,2}, G_{,3}) = 1$ to compute $\widetilde{G}\widetilde{H_3}$, and hence with the result from step 1 obtain $\widetilde{G}$;
3. the conservation laws arising from the symmetry X_T (12) and the multiplier $\mathrm{M}^{(7)}[G]$ (67) with $F(G_{,1}, G_{,2}, G_{,3}) = G_{,1}$, $G_{,2}$, and $G_{,3}$, respectively, yield $\widetilde{G_{,1}}\widetilde{H_3}$, $\widetilde{G_{,2}}\widetilde{H_3}$ and $\widetilde{G_{,3}}\widetilde{H_3}$, and hence $\widetilde{G_{,1}}$, $\widetilde{G_{,2}}$ and $\widetilde{G_{,3}}$;
4. finally, consider the six conservation laws arising from the symmetry X_T (12) and each of the six multipliers $\mathrm{M}^{(1)}[G], \ldots, \mathrm{M}^{(6)}[G]$ given by (65), (66) and their cyclic permutations. This yields the six linearly independent quantities (right-hand sides of (65), (66) and their cyclic permutations) on the next time step. Using the knowledge of $\widetilde{G}$, $\widetilde{G_{,1}}$, $\widetilde{G_{,2}}$ and $\widetilde{G_{,3}}$, one can solve the six equations to obtain the values of the second derivatives $\widetilde{G_{,11}}$, $\widetilde{G_{,12}}$, $\widetilde{G_{,13}}$, $\widetilde{G_{,22}}$, $\widetilde{G_{,23}}$, and $\widetilde{G_{,33}}$ on the next time step.

Similarly to the 2D case, if only G and its first spatial derivatives are required, one can simplify the algorithm accordingly. The near-singular behaviour is also avoided analogously to the 2D case.

Note that in both the 2D and 3D cases the numerical algorithm is based on six and 10 conservation laws, respectively. Hence, the computational effort of traditionally advancing a single level-set equation compared to several equations in the present procedures is considerably increased. Still, it might be very well worth the effort because of three reasons:

(i) a broad variety of robust and efficient numerical schemes for hyperbolic conservation laws have been developed which may directly be applicable. Hence, near singular behavior and even weak solutions are readily computable;
(ii) as a result the possible numerical instability when advancing the G-equation alone and hence the rather time consuming re-initialization method may be avoided;
(iii) even the case $\boldsymbol{u} \neq 0$ may be accessible by way of an operator splitting method though somewhat different from classical Strang or Lie splitting.

In essentially all splitting methods a PDE system with the dependent variable Γ of the form

$$\frac{\partial \Gamma}{\partial t} = \mathcal{L}_1[\Gamma] + \mathcal{L}_2[\Gamma] \tag{70}$$

with $\mathcal{L}_i$ are spatial operators, is splitted into at least two sets of equations with a single spatial operator

$$\frac{\partial \Gamma}{\partial t} = \mathcal{L}_1[\Gamma] \quad \text{and} \quad \frac{\partial \Gamma}{\partial t} = \mathcal{L}_2[\Gamma], \tag{71}$$

which are numerically evaluated by some alternating scheme. We now split (1) into

$$\frac{\partial G}{\partial t} = -u_k \frac{\partial G}{\partial x_k} \Leftrightarrow \frac{\partial G}{\partial t} = -\frac{\partial (u_k G)}{\partial x_k}, \tag{72}$$

where for the second form the continuity equation $\nabla \cdot \boldsymbol{u} = 0$ has been applied, and

$$\frac{\partial G}{\partial t} = s_l \sqrt{\frac{\partial G}{\partial x_k}\frac{\partial G}{\partial x_k}}. \tag{73}$$

Since (72b) is already in conserved form it is computationally accessible by any of the above mentioned schemes for hyperbolic conservation laws.

For the numerical evaluation of (73), the conserved forms derived in Sect. 3 can be used. Hence the numerical procedures based on the conserved forms pointed out above are directly applicable.

The actual time advancement of the full G-equation has to be based on a new algorithm since classical splitting schemes such as Strang or Lie splitting are not directly applicable. This is because the quantity to be advanced in time is not the same as in the separated equations as, e.g., in (71). Instead, here in the one step G alone is advanced due to (72b) while in the second step (73) will be advanced due to the above given conserved forms. In order to perform this in an alternating procedure, intermediate values for G and its derivatives may be re-computed in order to evaluate, e.g., $\mathrm{M}^{(i)}[G]$ in the density and fluxes of a conservation law (60).

If only G and its first spatial derivatives are to be computed, the above procedure for computing G-equation alone with $\boldsymbol{u} = 0$ has to be slightly modified, since quantities such as $\mathrm{M}^{(i)}[G]$, which also contain second-order derivatives in G, need to be re-initialized (re-computed) after an intermediate step where (72) has been conducted. This, of course, is not necessary when the G-equation alone is computed with $\boldsymbol{u} = 0$ since there second order derivatives of G needs only be evaluated from the initial G-field.

It is well-known that flame fronts described by the G-equation often develop cusps (corners). Therefore in the whole flame domain, one should generally expect spatial partial derivatives of G to be discontinuous. In particular, if spatial partial derivatives $G_t, G_{,i} \in L^1_{loc}(\mathbb{R}^n)$, then second spatial derivatives are *generalized functions* (distributions). The product of a discontinuous function and a distribution, in general, may not be consistently defined.[1] It follows that for conservation laws of the G-equation arising from multipliers, in order to consider globally non-smooth solutions, it is necessary to restrict conservation law multipliers $\Lambda[G]$ to belong to the class $L^1_{loc}(\mathbb{R}^n)$, i.e., $\Lambda[G] = \Lambda(t, \boldsymbol{x}, G, \partial G)$. However, no conservation laws arise from such ansatz; see Sect. 3. In fact, all conservation laws of the G-equation in 2D and 3D space found in Sect. 3 arise from multipliers essentially involving second derivatives of G, and therefore can be used in numerical methods either only locally in the domains of smoothness of G, or possibly in numerical methods that involve splitting.

4.2 Numerical schemes based on geometrical properties

A rather straightforward alternative idea for advancing the G-equation is based on solving the G-equation extended by an arbitrary order of geometrical quantities such as the normal vector and curvature measures. These additional equations emerge either directly from the G-equation or, more naturally, from invariant quantities pointed out in Sect. 2.2, which arise from the higher-order symmetries discussed therein, and propagate on the G level-set.

In the simplest possible procedure, which goes beyond solving the G-equation alone, the coupled system consists of

$$\frac{\mathrm{d}G}{\mathrm{d}t} = 0, \tag{74}$$

[1] A simple example is given by $u(x) = \{-1, x < 0;\ 1, x \geq 0\}$. One finds $u'(x) = 2\delta(x)$, where $\delta(x)$ is the Dirac delta function. By symmetry, it is natural to define $u(x)\delta(x) = 0$. Consider $u^2(x)\delta(x)$. On one hand, $u^2(x)\delta(x) = u(x)(u(x)\delta(x)) = 0$. On the other hand, one has $u^2(x)\delta(x) = (u(x)u(x))\delta(x) = \delta(x) \neq 0$. Thus for the pair of functions $(u(x), \delta(x))$, multiplication is not associative.

and

$$\frac{\mathrm{d}n_i}{\mathrm{d}t} = -\frac{\partial u_k}{\partial x_i} n_k + \frac{\partial u_k}{\partial x_l} n_k n_l n_i, \tag{75}$$

where the Lagrange-like operator is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + (u_k + s_l n_k) \frac{\partial}{\partial x_k}. \tag{76}$$

Note that in the numerical solution, G and $\boldsymbol{n}$ are considered independent variables.

If higher-order accuracy is intended, the system may be further extended by the equation for second derivatives $G_{,ij}$, as follows

$$\frac{\mathrm{d}G_{,ij}}{\mathrm{d}t} = -\frac{\partial^2 u_k}{\partial x_i \partial x_j} G_{,k} - \frac{\partial u_k}{\partial x_i} G_{,kj} - \frac{\partial u_k}{\partial x_j} G_{,ki} + s_l \left(\frac{G_{,im} G_{,jm}}{\sqrt{G_{,n} G_{,n}}} - \frac{G_{,im} G_{,m} G_{,jn} G_{,n}}{(G_{,k} G_{,k})^{3/2}} \right). \tag{77}$$

Similarly, one may also consider equations for even higher orders of spatial derivatives of G.

5 Discussion

The two pivotal goals of the present paper have been to compute higher-order symmetries for the G-equation with arbitrary flow velocity and to derive local conservation laws. Symmetries for the G-equation with zero flow velocity were already exhaustively computed up to the order two in [19]. We have shown that for an arbitrary flow velocity only the reduction to a system of ODE is conducted and hence only a formal procedure has been derived. A complete solution was not computed. Instead for a given flow velocity the symmetries may be computed by solving the latter ODE system.

Second, it has been shown that for arbitrary flow velocity there exists no conservation law for the G-equation. Instead it was discovered that an infinite set of conservation laws exist for the G-equation with constant flow velocity, or, without loss of generality, for zero flow velocity. For the two-dimensional case the complete set of multipliers were computed up to the order two. From this and the infinitesimals the conserved densities and fluxes have been derived. For the three-dimensional case only a reduced set of multipliers were computed because of its computational complexity.

Based on the latter, novel algorithms for the numerical solution of the G-equation (1) based on local symmetry and conservation law structure of the G-equation have been proposed. The numerical properties of these schemes such as accuracy or scheme stability will be investigated in subsequent research.

In particular, the scheme based on local conservation laws of the G-equation (Sect. 4.1) is rather different from classical splitting schemes, and its accuracy and stability will depend on the algorithms employed for solving the sub-equations (72) and (73) in conserved form.

For the symmetry-based numerical scheme (Sect. 4.2), the actual numerical implementation can be chosen from a broad variety of schemes, either based on field equations, such as finite-volume and finite-difference schemes, or based on geometrical concepts such as ray-tracing algorithms. Combinations of such approaches are also possible.

Several advantages and disadvantages of the proposed symmetry-based numerical scheme may be observed. The entire system may be solved using classical field methods with level-set specific ideas (such as the extension velocity method) built in. Hence any existing scheme for the G-equation may readily be extendable. Since the system (74) and (75) is polynomial, no near-singular behavior is to be expected for short times. Still, if strong topology changes are expected, a re-initialization scheme may be employed to avoid numerical instabilities. Also, it should be observed that there is a weak coupling, i.e., one-way coupling between (74) and (75).

Finally, two important things need to be pointed out. First, both numerical procedures described in Sect. 4 may need a flame-front reconstruction algorithm, if the location of the flame front is of interest. Second, additional schemes are needed if the G-equation is solved in conjunction with some fluid-dynamic transport equations. In

the majority of practical applications, a density jump at the flame front needs to be accounted for, and hence complementary algorithms need to be employed such as cut-cell methods.

In future research, it intended to perform numerical simulations for both of the above schemes, using a parallel discontinuous Galerkin code that is currently being developed. In terms of further analysis of the G-equation (1), it is of interest to classify all second-order conservation laws for 2D and 3D cases, and to seek higher-order conservation laws, in particular, in the case of nonzero flow velocity.

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