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Analysis

Constant per capita consumption paths with exhaustible resources and decaying produced capital

Alexei F. Cheviakov [a], John Hartwick [b,*]

[a] Department of Mathematics and Statistics, University of Saskatchewan, Saskatoon, Saskatchewan, Canada S7N 5E6
[b] Department of Economics, Queen's University, Kingston, Ontario Canada K7L 4L8

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ABSTRACT

We introduce decay in produced capital and exogenous technical progress to the recent "Solow Model" of Asheim et al. with population growth and observe the possible collapse of the economy given too high a rate of decay. "Enough" technical progress can restore sustainable per capita consumption.

1. Introduction

Solow (1974) suggested that sustainability for an economy in the face of depletion of an essential input such as oil for energy could be contemplated under a program of substituting produced capital in production for current use of oil capital. Essentially sustainability, non-declining per capita consumption, follows for an economy with "easy" substitution possibilities and asymptotic depletion of the stock of the essential depleting input. Mitra (1983) observed that the same degree of substitutability among inputs and asymptotic depletion was compatible with constant per capita consumption under the never-ending population growth. The economy was required to do "extra" savings for the case of positive population growth, "extra" being in excess of simply current , exhaustible resource rent. This striking case of Mitra-sustainability was re-visited by Asheim et al. (2007). There saving linear in output was linked directly to population growth of a quasi-arithmetic sort.[1] This latter sort of growth is less than exponential but remains unbounded.[2] Here we probe this model of ABHMW to examine cases of non-sustainability. When we introduce radio-active decay in produced capital, sustainability is not possible. We then consider technical progress as a fix and observe that indeed exogenous technical progress can restore sustainability, with both population growth and a positive decay rate for produced capital. Stiglitz (1974b) considered population growth and exhaustibility and left us with the impression that technical progress was necessary for the sustainability of per capita consumption. He however restricted his attention from the start to the case of exponential population growth. Our approach, following Mitra and ABHMW, is to start with a savings function and then solve for the population growth that works under constant per capita consumption. The population growth that works for our new model, with radio-active decay in produced capital and exogenous technical progress, is in the limit, exponential. The "allowable" population growth that we solve for has an exponent that depends on the parameters of the problem.

How should people uninterested in the technical details below react to these new sustainability results? If one is interested in issues of exhaustibility of say oil stocks one naturally turns first to the hard case of the input from the depleting stock being essential to current

* Corresponding author.
E-mail addresses: cheviakov@math.usask.ca (A.F. Cheviakov), hartwick@econ.queensu.ca (J. Hartwick).

[1] Mitra (2008) has cast the ABHMW is a quite general framework and establishes that quasi-arithmetic population growth is the best that a sustainable economy can do, the savings function being very general. He also obtains a form for sustainable savings, given a quite general population growth function. This new savings function he labels: a generalized Hartwick Rule since it resembles the "invest resource rents" rule in Hartwick (1977).

[2] $N(t) = \Gamma[A + Bt]^{\Omega}$ has the quasi-arithmetic form. For the case of growth, one usually has A, B, Γ and Ω positive.

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production. This restriction pretty well forces one to consider scenarios involving production with Cobb–Douglas production function. In addition to inputs being essential, the Cobb–Douglas case involves inputs in special substitutability relationships. What the recent extentions of the original Solow (1974) model are revealing is that "Cobb–Douglas substitutability" plus asymptotic depletion of an essential input from an exhaustible stock are compatible with some quite counter-intuitive outcomes, namely those involving per capita consumption remaining unchanging forever, even with population increasing in non-trivial ways. Our work here reminds us that a constant rate of decay in produced capital is a powerful "destroyer" of Solow sustainability and that one is obliged to turn to the familiar fix, namely exogenous technical progress in order to restore programs with constant per capita consumption. (This point was made recently by Cairns and Van Long (2006) in a setting with no population growth.) Stiglitz (1974b) established that exponential population growth was a powerful "destroyer" of sustainability and he invoked exogenous technical change to restore sustainability. Mitra (1983) and followers established that non-trivial population growth was in fact compatible with sustainability of the Solow (1974) sort and that a society was obliged to invest more than resource rents in produced capital if it were to be in a sustainable mode of development.

2. Exogenous technical progress and limiting exponential population growth

We start with the accounting relation, $C=Y-\dot{K}-\delta K$. Here C is an aggregate consumption flow, δ is the constant rate of decay in produced capital K, N is the labor force or flow input of labor to current production, $\dot{K}$ is the net investment or net additions to the current stock $K-\delta K$, and $Y=Y(t, K, R, N)$ is a production function. The function $R=R(t)$ is an extraction rate of some finite stock $S(t)$ (e.g. oil): $R(t)=-\dot{S}(t)$, and the total used stock $S_0=\int_0^\infty R(t)dt$ is assumed to be a finite number not exceeding the total available stock S_{tot}.

Exogenous technical change proceeds at the constant rate θ. The basic account is then

$$C=Y(K,R,N,t)-\dot{K}-\delta K,\ Y(K,R,N,t)=e^{\theta t}F(K,R,N). \qquad (1)$$

For the Cobb–Douglas case, with constant returns to scale,

$$Y=e^{\theta t}K^{\alpha}R^{\beta}N^{1-\alpha-\beta}=\left[e^{\frac{\theta}{\alpha}t}K\right]^{\alpha}R^{\beta}N^{1-\alpha-\beta}$$

and we will see this term $\frac{\theta}{\alpha}$ as below central. $\frac{\theta}{\alpha}$ is then the rate of capital augmenting technical change and $\frac{\theta}{\alpha}$ appears centrally in our condition below on "allowable" rates of decay δ for produced capital. Roughly speaking then δ and $\frac{\theta}{\alpha}$ are countering each other's impact on the growing economy. In per capita terms, we define $n=n(t)=\dot{N}/N$, $k=\frac{K}{N}$, $r=\frac{R}{N}$, and $y=Y/N=e^{\theta t}k^{\alpha}r^{\beta}$, and obtain

$$c=y-\dot{k}-k[n+\delta]. \qquad (2)$$

We adopt a savings rule, linear in current output (following ABHMW (2007)):

$$\dot{K}+\delta K=[\beta+\gamma]Y,\ \gamma=const. \qquad (3)$$

βY is the resource rent and γY is the supplementary or additional savings. Consumption is then given by $C=[1-\beta-\gamma]Y$, and thus the requirement of constant per capita consumption $c=C/N=\text{const}$ implies

$$\frac{\dot{Y}}{Y}-\frac{\dot{N}}{N}=0. \qquad (4)$$

Dynamic efficiency in extraction satisfies (Hotelling Rule):

$$\frac{\dot{Y}}{Y}-\frac{\dot{R}}{R}=\frac{\alpha Y}{K}. \qquad (5)$$

Thus the current model of sustainable per capita consumption with an essential exhaustible resource and exogenous technical progress is given by three nonlinear ordinary differential equations (ODE): Eqs. (3), (5), and (4), for three unknown functions $K=K(t)$, $R=R(t)$ and $N=N(t)$. It turns out that one is able to obtain *the general solution* of this nonlinear ODE system, as follows.

Since $\frac{\dot{Y}}{Y}=\theta+\alpha\frac{\dot{K}}{K}+\beta\frac{\dot{R}}{R}+(1-\alpha-\beta)\frac{\dot{N}}{N}$, we use Eqs. (3)–(5) to obtain

$$\frac{\dot{N}}{N}=\left(\frac{\theta}{\alpha}-\delta\right)+\frac{\gamma}{\alpha}\left(\frac{\alpha Y}{K}\right), \qquad (6)$$

$$\frac{\dot{R}}{R}=\left(\frac{\theta}{\alpha}-\delta\right)+\frac{\gamma-\alpha}{\alpha}\left(\frac{\alpha Y}{K}\right), \qquad (7)$$

$$\frac{\dot{K}}{K}=\frac{\beta+\gamma}{\alpha}\left(\frac{\alpha Y}{K}\right)-\delta. \qquad (8)$$

In per capita terms, we obtain

$$n=\left(\frac{\theta}{\alpha}-\delta\right)+\frac{\gamma}{\alpha}\left(\frac{\alpha y}{k}\right), \qquad (9)$$

$$\dot{k}=\beta y-\frac{\theta k}{\alpha}, \qquad (10)$$

$$\frac{\dot{y}}{y}-\frac{\dot{r}}{r}=\frac{\alpha y}{k}. \qquad (11)$$

In the savings rule Eq. (10) βy is the per capita resource rent. For ABHMW, the corresponding savings rule reduced to $\beta y=\dot{k}$. "Supplementary savings" was then $\gamma y=kn$. Our supplementary savings are $\gamma y=kn+\delta k-\frac{\theta}{\alpha}k$. For Solow (1974) $\gamma y=0$.

The two equations, namely Eqs. (10) and (11), imply that $\dot{y}=0$ (which follows from substitution into $\frac{\dot{y}}{y}=\theta+\alpha\frac{\dot{k}}{k}+\beta\frac{\dot{r}}{r}$.) The general solution of the coupled nonlinear ODE system of Eqs. (10) and (11) is

$$k=k(t)=Ae^{-\frac{\theta}{\alpha}t}+\frac{\alpha\beta y}{\theta}, \qquad (12)$$

$$r=r(t)=B[k(t)]^{-\frac{\alpha}{\beta}}e^{-\frac{\theta}{\beta}t}, \qquad (13)$$

where A, $B=\text{const}$ are two constants of integration, and

$$y=e^{\theta t}k^{\alpha}(t)r^{\beta}(t)=B^{\beta}=const \qquad (14)$$

as expected. The long-term behaviour of this solution ($t\to\infty$) is

$$k\to\frac{\alpha\beta y}{\theta},\ r\to 0,\ \frac{\dot{r}}{r}\to-\frac{\theta}{\beta}=const.$$

Given the solution in Eqs. (12) and (13), we observe that the population dynamics of Eq. (6) integrates yielding

$$N = N(t) = N_0[k(t)]^{\frac{\gamma}{\beta}} e^{\left[\frac{\theta}{\alpha}\left(1+\frac{\gamma}{\beta}\right)-\delta\right]t} \tag{15}$$

with $k(t)$ given by Eq. (12). Since $k(t) \to$ const as $t \to \infty$, the asymptotic behaviour of such population is either exponential growth or exponential decay, depending on the sign of the expression $\frac{\theta}{\alpha}\left(1+\frac{\gamma}{\beta}\right)-\delta$. From Eqs. (12),(13), and (15) we readily find the input capital $K(t) = N(t)k(t)$ and oil extraction flow $R(t) = N(t)r(t)$ to be

$$K = K(t) = N_0[k(t)]^{\frac{\gamma}{\beta}+1} e^{\left[\frac{\theta}{\alpha}\left(1+\frac{\gamma}{\beta}\right)-\delta\right]t}, \tag{16}$$

$$R = R(t) = N_0 B[k(t)]^{\frac{\gamma-\alpha}{\beta}} e^{\left[\frac{\theta}{\alpha}\left(1+\frac{\gamma-\alpha}{\beta}\right)-\delta\right]t}. \tag{17}$$

We assume that $\alpha > \gamma$. This figures somewhat tangentially below. The scenario compatible with a finite initial stock of oil requires that $0 < \int_0^\infty R(t)dt < +\infty$. Since $k(t)$ is bounded, from Eq. (17) it follows that the oil stock is finite if and only if

$$\frac{\theta}{\alpha}\left(1+\frac{\gamma-\alpha}{\beta}\right)-\delta < 0.$$

Since $k(t)$ tends to a constant in the limit, for growth in the economy (i.e. for $N(t)$ and $K(t)$ increasing) we require that the factor in the exponent, namely

$$\frac{\theta}{\alpha}\left(1+\frac{\gamma}{\beta}\right)-\delta > 0.$$

We have then an upper and lower bound on a value of δ such that the economy is feasible and expanding[3]; namely

$$\delta_1^* < \delta < \delta_2^*, \tag{18}$$

where

$$\delta_1^* = \frac{\theta}{\alpha}\left(1+\frac{\gamma-\alpha}{\beta}\right), \; \delta_2^* = \frac{\theta}{\alpha}\left(1+\frac{\gamma}{\beta}\right).$$

The intuition here is that a sufficiently high value of decay, δ, induces a drag in the economy in order to keep long-run production within the given initial resource stock, but too large a value of δ rules out output growth compatible with the population growing, as opposed to shrinking. The lower bound on δ namely δ_1^* will decline for $\frac{\theta}{\alpha}$ smaller (less technical progress) or γ smaller (less supplementary savings). Roughly speaking a δ "too small" implies "too rapid" an accumulation of $K(t)$ relative to the availability of natural resources.

2.1. Examples of the solution curves for various initial conditions

In Fig. 1, we show examples of solutions (15), (16), (17) that exhibit exponential decay ($\delta > \delta_2^*$) and exponential growth ($\delta < \delta_2^*$). The constant values used are $\alpha = 0.65$, $\beta = 0.1$, $\gamma = 0.23$, $\theta = 0.0025$; the initial conditions are $N(0) = R(0) = 1$, $K_0 = 1000$. (Our parameter values have not been selected to approximate the stylized facts.) Solution curves are plotted for $\delta = 0.0015$; $\delta = \delta_1^* \simeq 0.0046$; $\delta = 0.017$; $\delta = \delta_2^* \simeq 0.0296$; $\delta = 0.0444$. Thus the first solution curve corresponds to infinite stock oil; third solution curve corresponds to Eq. (18) (exponentially growing population with finite oil stock), fifth solution curve corresponds $\delta > \delta_2^*$ (exponentially shrinking population with finite oil stock). Solution curve two corresponds to the boundary case $\delta = \delta_1^*$ with constant rate of oil extraction $R(t) =$ const and growing population. Solution curve four corresponds to the boundary case $\delta = \delta_2^*$ with finite oil stock and constant population: $N(t) =$ const, $K(t) =$ const.

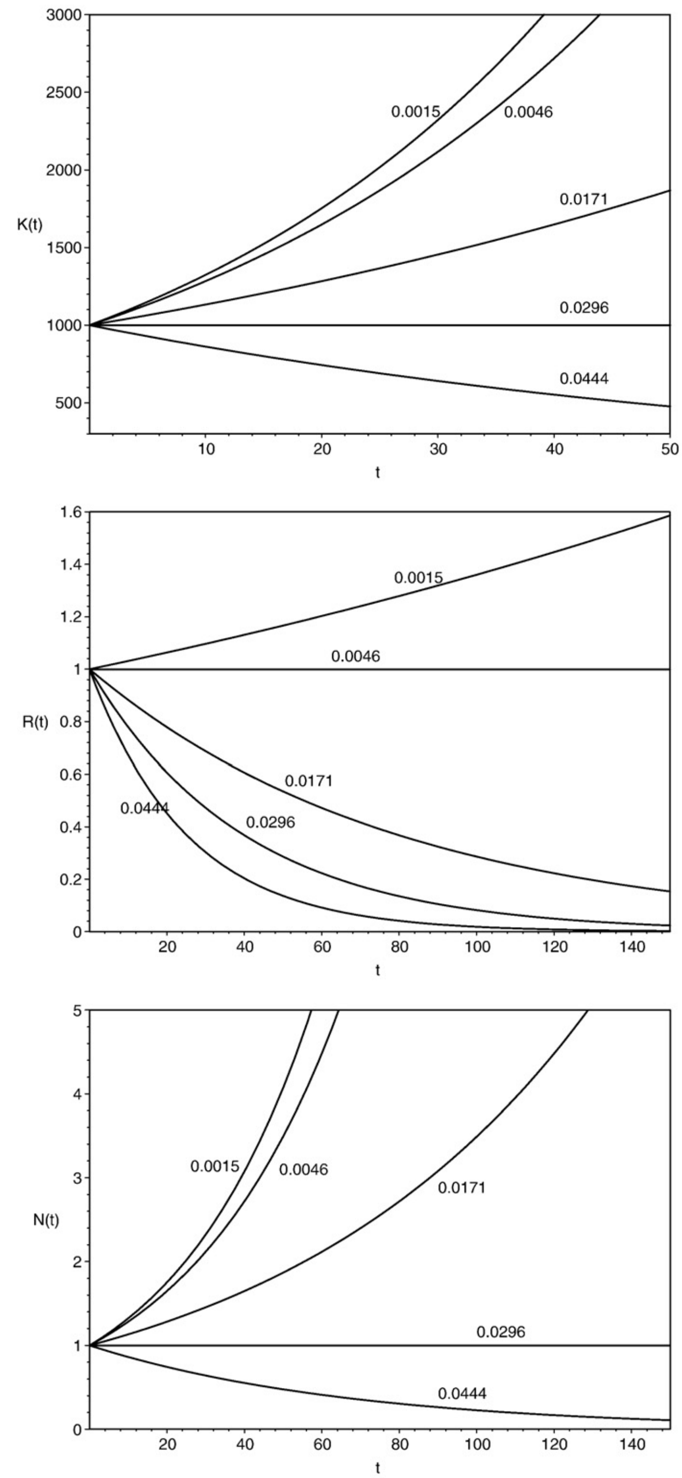

Fig. 1. Profiles of $K(t)$, $R(t)$ and $N(t)$ of the solution (15), (16), (17) of the model of sustainable consumption with exogenous technical progress (Section 2). Here $\alpha = 0.65$, $\beta = 0.1$, $\gamma = 0.23$, $\theta = 0.0025$. The initial conditions are $N(0) = R(0) = 1$, $K_0 = 1000$. Curves are given for values of $\delta = (0.0015;\ \delta_1^* \simeq 0.0046;\ 0.017;\ \delta_2^* \simeq 0.0296;\ 0.0444)$. This range of δ includes cases of infinite and finite oil stock ($\delta \lesseqgtr \delta_1^*$), growing and shrinking population ($\delta \lesseqgtr \delta_2^*$), as well as boundary cases $\delta = \delta_1^*$, δ_2^*.

[3] The economy will be feasible and contracting when $\delta > \frac{\theta}{\alpha}\left\{\frac{\beta+\gamma}{\beta}\right\}$.

3. Population growth exponential and exogenous

Consider the special case of n constant (Stiglitz (1974a)). This indicates that k is constant in the per capita version of the savings rule (3):

$$\gamma y = [n+\delta]k - \frac{\theta k}{\alpha}$$

in addition to $y=$const. Hence we have

$$\gamma \frac{y}{k} = (n+\delta) - \frac{\theta}{\alpha}$$

and the condition

$$(n+\delta) - \frac{\theta}{\alpha} > 0.$$

We insert the expression for $\frac{y}{k}$ in Eq. (7) to observe that

$$\frac{\dot{R}}{R} = \frac{1}{\gamma}(n(\gamma-\alpha) - \alpha\delta + \theta).$$

This must be negative for sustainable consumption to be feasible in this "balanced growth" solution. For the Stiglitz case of $\delta=0$ and $n>0$, we thus require $(n(\gamma-\alpha)+\theta)$ negative or $\alpha>\gamma$ (noted by Stiglitz (1974a; Proposition 3)). Stiglitz's saddlepath is always achievable, given the correct initial value for $R(t)$.

4. No technical progress

We return to our original model above with population change endogenous. We specialize now to the case of no technical progress. This case is basically the one investigated in detail in Asheim et al. (2007) (i.e. $\delta=\theta=0$ in our original model.) The principal result in ABHMW was that supplementary savings linear in current output, γy "allows for" population growth with constant per capita consumption of a quasi-arithmetic form. The converse is also true: namely quasi-arithmetic population growth implies extra savings linear in current output. Quasi-arithmetic growth takes the form $N(t)=\zeta[A+Bt]^{\Delta}$ for ζ, Δ, A and B positive constants. The value of γ affects the rate of population growth via the value of Δ but not the form of the function. This is a major extension of accepted views of sustainability (e.g. Solow (1974)) since per capita consumption can be held constant forever, even with an ever increasing population and a finite stock of the essential resource input.

The interesting new special case of our general model above has $\theta=0$ but decay δ positive. This is an extension of ABHMW with a new non-trivial solution and a new behaviour, namely collapse for the economy. We have the same core system here (Eqs. (9)–(11) with $\theta=0$), yielding $\dot{y}=0$. We can re-solve the model above with both $n(t)$ and δ non-zero. The model is now

$$\dot{k} = \beta y, \tag{19}$$

$$\frac{\dot{r}}{r} = -\frac{\alpha y}{k}, \tag{20}$$

$$\frac{\dot{N}}{N} = n = -\delta + \frac{\gamma}{\alpha}\left(\frac{\alpha y}{k}\right). \tag{21}$$

Now extra savings γy covers nk plus δk. Recall that $\dot{k}=\beta k^{\alpha} r^{\beta}$ implies $\frac{\dot{K}}{K} = \frac{\beta Y}{K} + \frac{\dot{N}}{N} = \left[\frac{\beta+\gamma}{\alpha}\right]\frac{\alpha y}{k} - \delta$ and $\frac{\dot{r}}{r} = \frac{-\alpha k^{\alpha} r^{\beta}}{k}$ implies $\frac{\dot{R}}{R} = -\frac{\alpha y}{k} + \frac{\dot{N}}{N} = -\left[\frac{\alpha-\gamma}{\alpha}\right]\frac{\alpha y}{k} - \delta$. For the case of $N(t)$ decreasing, the rate of decline in $R(t)$ will be faster. One might interpret this as sort of conservation-oriented for oil.

Using $y=$const, the solution of ODEs (19), (20) is readily found to be

$$k = k(t) = b_1 + b_2 t, \tag{22}$$

$$r = r(t) = (\beta^{-1} b_2)^{1/\beta}[b_1 + b_2 t]^{-\alpha/\beta}, \tag{23}$$

The population dynamics (21) are therefore satisfied by

$$N(t) = Be^{-\delta t}(b_1 + b_2 t)^{\gamma/\beta}$$

for B a positive constant. This form for $N(t)$ implies exponential decline in $N(t)$ in the limit. That is, population must decline if constant consumption is to be maintained, with produced capital subject to radio-active decay at rate δ.

For oil use dynamics $R(t)=N(t)r(t)$ we have $R(t) = \left(\frac{b_2}{\beta}\right)^{1/\beta} Be^{-\delta t}(b_1 + b_2 t)^{\frac{\alpha-\gamma}{\beta}}$

Also $k(t)=N(t)k(t)$ yields

$$K(t) = Be^{-\delta t}(b_1 + b_2 t)^{1+\frac{\gamma}{\beta}}.$$

These solutions are very similar to those above for the Mitra (1983)–ABHMW model[4] with quasi-arithmetic population growth except that each has the new term, $e^{-\delta t}$. This latter implies decline at rate δ in the limit for K, N and R. This suggests that the level of per capita consumption sustainable in this economy will be zero.

5. Concluding remark

We have extended the recent work in Asheim et al. (2007) to come up with new interesting scenarios and new closed form solutions. Of interest is the sensitivity of the ABHMW model to a positive rate of decay for produced capital. Expansion turns to inevitable collapse. And we get a new picture of expansion in Stiglitz (1974b) with our addition of decay in produced capital. A fairly natural collapse scenario emerges as well as a somewhat richer range of possibilities for expansion in "balanced growth".

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[4] Mitra (1983) introduced quasi-arithmetic growth in the population into a discrete-time version of Solow (1974). Solow had no population increase. Mitra did not explicitly link this population increase to a savings function, the central focus of ABHMW.

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