On the Nonlocal Symmetries, Group Invariant Solutions and Conservation Laws of the Equations of Nonlinear Dynamical Compressible Elasticity

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Abstract A family of new PDE systems of one-dimensional nonlinear elastodynamics, which are nonlocally related to the classical Lagrange and Euler formulations, is derived. These new PDE systems provide alternative equivalent descriptions of the one-dimensional nonlinear elasticity model. In particular, nonlocally related systems are used to find nonlocal symmetries of the Euler system for various forms of constitutive and loading functions. Examples of new dynamical solutions arising as group invariant solutions with respect to such nonlocal symmetries are constructed. Another application of nonlocally related systems considered in this paper is the construction of nonlocal conservation laws. Examples of nonlocal conservation laws are derived for several classes of stress-strain relations and loading functions.

1 Introduction

Analytical studies of nonlinear dynamical elasticity models, and especially, the problem of finding exact solutions, have attracted significant attention of researchers in recent years. Lie symmetries are widely used in the analysis of contemporary nonlinear elasticity models, especially for the calculation of similarity (invariant) solutions, arising from symmetry reduction (see [1–4] and further references in [7]). Focusing on nonlinear elasticity, it is well known that very few closed-form solutions

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of BVPs for compressible elasticity have been obtained in the literature (contrary to incompressible elasticity), due to the absence of the kinematic incompressibility constraint, as pointed out in [8, 9]. Lie group analysis is of further interest in setting up numerical schemes preserving the group properties of an initial boundary value problem (BVP) [5].

The problem of finding conservation laws (full divergence expressions) that hold for a system of partial differential equations (PDEs) is an important counterpart to symmetry analysis. In elasticity theory, one well-known application of conservation laws is the Eshelby energy-momentum tensor, and a related path-independent integral, which governs the energy release rate at a singularity [10]. Symmetries play moreover an important role in Eshelbian mechanics, since invariance of a suitable functional under translations in material space (corresponding to horizontal variations) highlights the Eshelby tensor in the resulting equilibrium equation [6]. In a series of papers, Olver studied conservation laws and related path-independent integrals in linear elastostatics within the framework of hyperelasticity (see [11] and references therein). Mathematically, conservation laws can be systematically calculated, both for variational problems (Noether's theorem) and for non-variational problems [13, 14].

The present contribution is organized as follows. In Section 2, the Euler and Lagrange PDE systems of one-dimensional systems of nonlinear elastodynamics are presented. In Section 3, a nonlocal relation between the Euler and Lagrange PDE systems is derived, and using conservation laws of the Euler system, a set of additional PDE systems, equivalent but nonlocally related to both the Euler and Lagrange PDE systems, is constructed. In Section 4, nonlocal symmetries of the Euler system are classified, arising as point symmetries of its nonlocally related systems. Such nonlocal symmetries are used in Section 5 to derive an example of an exact invariant solution of a nonlinear elastodynamics BVP, corresponding to a nonlinear stretching. In Section 6, one of the considered nonlocally related PDE systems is used to derive new nonlocal conservation laws of the Euler equations of nonlinear elastodynamics.

2 Nonlinear Elasticity: Boundary Value Problems in 1D

In the one-dimensional situation, since the transformation gradient is the ratio of initial to actual density, $F = q = \rho_0/\rho$, the Cauchy stress is given by $\sigma = \sigma(\rho)$. The 1D Euler system is given by [7]

$$\mathbf{E}\{x, t\,;\, v, \sigma, \rho\} : \quad \begin{cases} \rho_t + (\rho v)_x = 0, \\ \sigma_x + \rho f(x,t) = \rho(v_t + v v_x), \\ \sigma = K(\rho). \end{cases} \tag{1}$$

In this paper we will only consider conservative forces $f(x,t) = f(x)$.

The independent variables are the Eulerian coordinates (x, t), the dependent variables are (ρ, v, σ), and some freedom of choice is allowed for the possible forms of the load per unit mass $f = f(x)$ and the material's constitutive response, i.e. the function $K = K(\rho)$. Dimensionless variables are adopted in the sequel, following [7].

The relationship between the first Piola–Kirchhoff stress and the Cauchy stress leads in 1D to $\sigma = T$. Therefore, the 1D Lagrange system in dimensionless variables is given by

$$\mathbf{L}\{y, s\,;\, v, \sigma, q, x\} : \quad \begin{cases} q = x_y, \\ v = x_t, \\ v_t = \sigma_y + f(x), \\ \sigma = K(1/q). \end{cases} \tag{2}$$

In the Lagrange system, the independent variables are the Lagrangian coordinates (y, t), the dependent variables are (x, v, q, σ), and the free functions are $f = f(x)$ and $K_1(q) \equiv K(1/q)$.

3 Nonlocally Related Systems of 1D Nonlinear Elasticity

Consider a PDE system $\mathbf{R}\{x\,;\,u\}$ of N PDEs of order k with n independent variables $x = (x^1, \ldots, x^n)$ and m dependent variables $u(x) = (u^1(x), \ldots, u^m(x))$, given by

$$R^\sigma[u] \equiv R^\sigma(x, u, \partial u, \ldots, \partial^k u) = 0, \quad \sigma = 1, \ldots, N. \tag{3}$$

PDE systems *nonlocally related* to $\mathbf{R}\{x\,;\,u\}$ arise in the following two simple ways [12].

(a) If the system $\mathbf{R}\{x\,;\,u\}$ has a *conservation law*

$$D_i \Phi^i[u] = 0, \tag{4}$$

then one may accordingly introduce nonlocal (potential) variable(s) v, satisfying corresponding potential equations. The union of the set of equations of $\mathbf{R}\{x\,;\,u\}$ and the potential equations yields a *potential system* $\mathbf{S}\{x\,;\,u, v\}$.

(b) Exclusion of one of the dependent variables of $\mathbf{R}\{x\,;\,u\}$ by differential compatibility relations (e.g. $v_{xt} = v_{tx}$) yields a *nonlocally related subsystem*. For example, if u^1 can be excluded, the corresponding subsystem is denoted $\underline{\mathbf{R}}\{x\,;\,u^2, \ldots, u^n\}$.

Combinations of the above two constructions, including their use in combination with interchanges of dependent and independent variables, may be used to obtain further nonlocally related PDE systems.

Solution sets of nonlocally related PDE systems are *equivalent*, in the sense that the solution set of one such system can be found from the solution set of of any other one. Therefore any method of analysis (qualitative, perturbation, numerical, etc.) that fails to work for a given PDE system, especially a method that is not coordinate-dependent, could turn out to be successful when applied to such a nonlocally related PDE system. In particular, for a given PDE system, through Lie's algorithm applied to a nonlocally related system, one can systematically calculate *nonlocal symmetries* (which in turn are useful for obtaining new exact solutions from known ones), construct (further) invariant and nonclassical solutions, as well as obtain linearizations (see e.g. [7, 12]). One could also obtain *nonlocal conservation laws* of a given PDE system, through the application of a standard procedure for finding local conservation laws to a nonlocally related system (see Section 6).

We now construct a tree of nonlocally related systems for 1D nonlinear elastodynamics, starting with the Euler system $\mathbf{E}\{x, t\,;\, v, \sigma, \rho\}$ (1).

The first equation of the system $\mathbf{E}\{x, t\,;\, v, \sigma, \rho\}$ (1) is in the form of a conservation law (mass conservation) as it stands; hence a potential w can be introduced. The corresponding potential system takes the form

$$\mathbf{EW}\{x, t\,;\, v, \sigma, \rho, w\} : \quad \begin{cases} w_x = \rho, \\ w_t = -\rho v, \\ \sigma_x + \rho f(x) = \rho(v_t + v v_x), \\ \sigma = K(\rho). \end{cases} \tag{5}$$

It is remarkable that a *local* 1:1 point transformation (an interchange of a dependent and independent variable) of the system $\mathbf{EW}\{x, t\,;\, v, \sigma, \rho, w\}$ with $w = y$ and t treated as independent variables, and $x, v, \sigma, q = 1/\rho$ as dependent variables, directly yields the Lagrange system $\mathbf{L}\{y, s\,;\, v, \sigma, q, x\}$ (2). Hence, the systems $\mathbf{EW}\{x, t\,;\, v, \sigma, \rho, w\}$ (5) and $\mathbf{L}\{y, s\,;\, v, \sigma, q, x\}$ (2) are *locally related to each other* (by a point transformation), but *nonlocally related to the Euler system* $\mathbf{E}\{x, t\,;\, v, \sigma, \rho\}$ (1). A similar connection exists in higher dimensions, expressed by the kinematic relation from configurational mechanics given by

$$y_{,t} + \mathbf{F}^{-1}.x_{,t} = 0$$

In the Lagrange system $\mathbf{L}\{y, s\,;\, v, \sigma, q, x\}$ (or $\mathbf{EW}\{x, t\,;\, v, \sigma, \rho, w\}$), the independent variable $y = w = \int \rho(x, t)dx$ is a *mass coordinate*.

Note that in the case of *linear elastodynamics*, $\sigma = K(\rho) = \rho_0/\rho$ with linear loading $f(x)$, the system $\mathbf{EW}\{x, t\,;\, v, \sigma, \rho, w\}$ (5) is a nonlinear PDE system, whereas the locally equivalent system $\mathbf{L}\{y, s\,;\, v, \sigma, q, x\}$ (2) becomes *linear*.

To further extend the tree of nonlocally related systems of one-dimensional nonlinear elasticity equations, one can use additional conservation laws and consider potential systems of the PDE systems $\mathbf{E}\{x, t\,;\, v, \sigma, \rho\}$, $\mathbf{EW}\{x, t\,;\, v, \sigma, \rho, w\}$ and/or $\mathbf{L}\{y, s\,;\, v, \sigma, q, x\}$. In particular, the Euler system $\mathbf{E}\{x, t\,;\, v, \sigma, \rho\}$ (1) has the conservation law

$$D_t(v - f(x)t) + D_x\left(\frac{v^2}{2} - M(\rho)\right) = 0, \tag{6}$$

where $M(\rho) = \int \frac{K'(\rho)}{\rho} d\rho$. Introducing a potential variable $r(x,t)$, one obtains the potential equations

$$r_x = v - f(x)t; \quad r_t = M(\rho) - \frac{v^2}{2} \tag{7}$$

The Euler system $\mathbf{E}\{x, t\,;\, v, \sigma, \rho\}$ (1) also has a conservation law corresponding to the conservation of energy:

$$D_t\left(\rho\frac{v^2}{2} - \int M(\rho)d\rho - \rho\int f(x)dx\right) + D_x\left(\rho v\left[\frac{v^2}{2} - M(\rho) - \int f(x)dx\right]\right) = 0, \tag{8}$$

which yields the potential equations

$$s_x = \rho\frac{v^2}{2} - \int M(\rho)d\rho - \rho\int f(x)dx; \quad s_t = -\rho v\left[\frac{v^2}{2} - M(\rho) - \int f(x)dx\right]. \tag{9}$$

The nonlocal variable

$$s(x,t) = \int\left(\rho\frac{v^2}{2} - \int M(\rho)d\rho - \rho\int f(x)dx\right)dx$$

is an “energy coordinate", analogous to the mass coordinate w and the “velocity coordinate" r. The three conservation laws (mass, average velocity and energy) yield the following seven distinct nonlocally related (potential) systems of the Euler system $\mathbf{E}\{x, t\,;\, v, \sigma, \rho\}$ [12]:

- Three *singlet* potential systems: $\mathbf{EW}\{x, t\,;\, v, \sigma, \rho, w\}$ (5),

$$\mathbf{ER}\{x, t\,;\, v, \sigma, \rho, r\}: \quad \begin{cases} \rho_t + (\rho v)_x = 0, \\ r_x = v - f(x)t, \\ r_t = M(\rho) - v^2/2, \\ \sigma = K(\rho). \end{cases} \tag{10}$$

  and

$$\mathbf{ES}\{x, t\,;\, v, \sigma, \rho, s\}: \quad \begin{cases} \rho_t + (\rho v)_x = 0, \\ s_x = \rho v^2/2 - \int M(\rho)d\rho - \rho\int f(x)dx, \\ s_t = -\rho v\left[v^2/2 - M(\rho) - \int f(x)dx\right], \\ \sigma_x + \rho f(x) = \rho(v_t + v v_x), \\ \sigma = K(\rho). \end{cases} \tag{11}$$

- Three *couplet* potential systems:

$$\mathbf{ERW}\{x,t\,;\,v,\sigma,\rho,r,w\}:\quad \begin{cases} \rho_t+(\rho v)_x=0,\\ r_x=v-f(x)t,\\ r_t=M(\rho)-v^2/2,\\ w_x=\rho,\\ w_t=-\rho v,\\ \sigma=K(\rho). \end{cases} \tag{12}$$

$$\mathbf{ESW}\{x,t\,;\,v,\sigma,\rho,s,w\}:\quad \begin{cases} \rho_t+(\rho v)_x=0,\\ s_x=\rho v^2/2-\displaystyle\int M(\rho)d\rho-\rho\int f(x)dx,\\ s_t=-\rho v\left[v^2/2-M(\rho)-\displaystyle\int f(x)dx\right],\\ w_x=\rho,\\ w_t=-\rho v,\\ \sigma_x+\rho f(x)=\rho(v_t+vv_x),\\ \sigma=K(\rho). \end{cases} \tag{13}$$

and

$$\mathbf{ERS}\{x,t\,;\,v,\sigma,\rho,r,s\}:\quad \begin{cases} \rho_t+(\rho v)_x=0,\\ r_x=v-f(x)t,\\ r_t=M(\rho)-v^2/2,\\ s_x=\rho v^2/2-\displaystyle\int M(\rho)d\rho-\rho\int f(x)dx,\\ s_t=-\rho v\left[v^2/2-M(\rho)-\displaystyle\int f(x)dx\right],\\ \sigma_x+\rho f(x)=\rho(v_t+vv_x),\\ \sigma=K(\rho). \end{cases} \tag{14}$$

- One *triplet* potential system

$$\mathbf{ERSW}\{x,t\,;\,v,\sigma,\rho,r,s,w\}:\quad \begin{cases} \rho_t+(\rho v)_x=0,\\ r_x=v-f(x)t,\\ r_t=M(\rho)-v^2/2,\\ s_x=\rho v^2/2-\displaystyle\int M(\rho)d\rho-\rho\int f(x)dx,\\ s_t=-\rho v\left[v^2/2-M(\rho)-\displaystyle\int f(x)dx\right],\\ w_x=\rho,\\ w_t=-\rho v,\\ \sigma_x+\rho f(x)=\rho(v_t+vv_x),\\ \sigma=K(\rho). \end{cases} \tag{15}$$

(For $f(x)$ = const, one can obtain additional nonlocally related PDE systems, as discussed in [7].) Hence, *for arbitrary forms of the constitutive functions $K(\rho)$ and $f(x)$, one has a tree of equivalent and nonlocally*

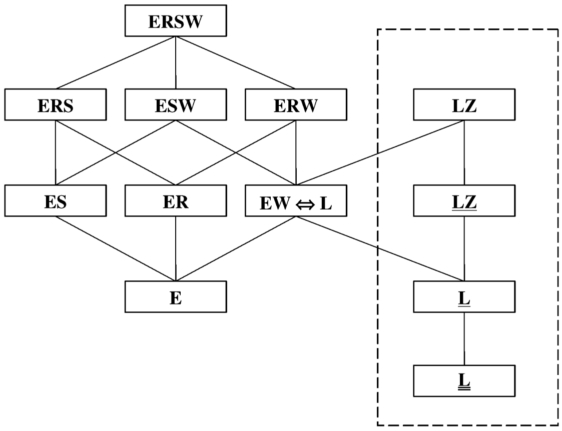

Fig. 1 Tree of nonlocally-related systems of nonlinear elasticity. (The dotted box corresponds to nonlocally related systems that arise for the case $f(x) = \text{const}$ [7].)

related systems of nonlinear elasticity, consisting of eight PDE systems $\mathbf{E}\{x, t\,;\, v, \sigma, \rho\}$, $\mathbf{L}\{y, s\,;\, v, \sigma, q, x\} \Leftrightarrow \mathbf{EW}\{x, t\,;\, v, \sigma, \rho, w\}$, $\mathbf{ER}\{x, t\,;\, v, \sigma, \rho, r\}$, $\mathbf{ES}\{x, t\,;\, v, \sigma, \rho, s\}$, $\mathbf{ERW}\{x, t\,;\, v, \sigma, \rho, r, w\}$, $\mathbf{ESW}\{x, t\,;\, v, \sigma, \rho, s, w\}$, $\mathbf{ERS}\{x, t\,;\, v, \sigma, \rho, r, s\}$, *and* $\mathbf{ERSW}\{x, t\,;\, v, \sigma, \rho, r, s, w\}$ (Figure 1). All nonlocally related PDE systems in the tree provide equivalent descriptions of nonlinear 1D elastodynamics, and thus naturally extend the traditional Lagrangian and Eulerian viewpoints.

4 Point and Nonlocal Symmetry Classification of the Lagrange System $\mathbf{EW}\{x, t\,;\, v, \sigma, \rho, w\} \Leftrightarrow \mathbf{L}\{y, s\,;\, v, \sigma, q, x\}$

A *symmetry* of a system of PDEs is any transformation of its solution manifold into itself (i.e., a symmetry transforms any solution to another solution of the same system).

Lie's algorithm is used to find one-parameter (ε) Lie groups of point transformations (point symmetries)

$$\begin{aligned}(x^*)^i &= f^i(x, u; \varepsilon), \quad i = 1, \dots, n,\\ (u^*)^j &= g^j(x, u; \varepsilon), \quad j = 1, \dots, m,\end{aligned} \tag{16}$$

that leave invariant a given system of N partial differential equations $\mathbf{R}\{x; u\}$ [1–4] such that $R^\rho[u^*] = 0$, $\rho = 1, \dots, N$, if and only if $R^\sigma[u] = 0$, $\sigma = 1, \dots, N$.

Global Lie transformation groups (16) are in one-to-one correspondence with local transformations

$$\begin{aligned}(x^*)^i &= x^i + \varepsilon\xi^i(x, u) + O(\varepsilon^2), \quad i = 1, \dots, n,\\ (u^*)^j &= u^j + \varepsilon\eta^j(x, u) + O(\varepsilon^2), \quad j = 1, \dots, m,\end{aligned} \tag{17}$$

where ξ^i, η^j are components of a vector field (infinitesimal generator)

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^j(x,u)\frac{\partial}{\partial u^j} \tag{18}$$

tangent to the solution manifold of the given PDE system.

For a given PDE system, nonlocal symmetries called *potential symmetries* can arise naturally by applying Lie's algorithm to a related potential system. Such a symmetry is a nonlocal symmetry when at least one component of the symmetry generator has an essential dependence on a nonlocal variable. The complete point symmetry classification for the potential system $\mathbf{EW}\{x, t\,;\, v, \sigma, \rho, w\}$ (5) in terms of its constitutive and loading functions was presented in [7]. In particular, it was found that in the cases

$$K(\rho) = \frac{1}{2}\left(\arctan\frac{1}{\rho} + \frac{\rho}{\rho^2+1}\right) \quad \text{or} \quad K(\rho) = \frac{1}{4}\ln\frac{\rho-1}{\rho+1} - \frac{1}{2}\frac{\rho}{\rho^2-1},$$

for a linear body force $f(x) = x$, the potential system $\mathbf{EW}\{x, t\,;\, v, \sigma, \rho, w\}$ (5) has two point symmetries which are nonlocal symmetries of the Euler system $\mathbf{E}\{x, t\,;\, v, \sigma, \rho\}$ (1). One of these nonlocal symmetries is used in the following section to construct a corresponding exact invariant solution of the Euler system $\mathbf{E}\{x, t\,;\, v, \sigma, \rho\}$ (1).

5 Calculation of Group Invariant Solutions Arising from the Lagrange System EW$\{x, t\,;\, v, \sigma, \rho, w\}$ (5)

The general method for finding invariant solutions following from local symmetries is presented in detail in [1,4]. For invariant solutions arising from nonlocal (potential) symmetries, see also [15].

Let G be a one-parameter Lie group of point symmetries of the potential system $\mathbf{EW}\{x, t\,;\, v, \sigma, \rho, w\}$ (5), with an infinitesimal generator

$$X = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \eta^v\frac{\partial}{\partial v} + \eta^\rho\frac{\partial}{\partial \rho} + \eta^w\frac{\partial}{\partial w}. \tag{19}$$

Here $\xi, \tau, \eta^v, \eta^\rho$ and η^w are functions of x, t, v, σ, ρ and w. The corresponding invariant solutions

$$(v, \rho, w) \;=\; (V(x,t), R(x,t), W(x,t)) \tag{20}$$

of the potential system $\mathbf{EW}\{x, t\,;\, v, \sigma, \rho, w\}$ satisfy

$$X \cdot \begin{bmatrix} v - V(x,t) \\ \rho - R(x,t) \\ w - W(x,t) \end{bmatrix}\Bigg|_{(v,\rho,w)\,=\,(V(x,t),R(x,t),W(x,t))} = 0 \tag{21}$$

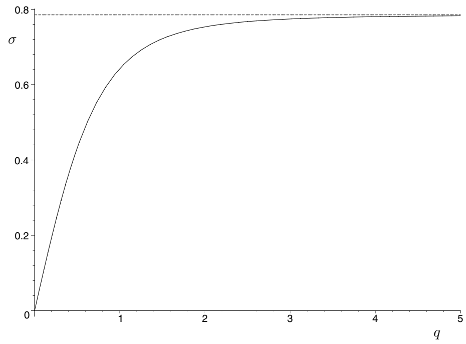

Fig. 2 Stress-strain curve for the constitutive relation $\sigma = K(\rho)$ given by (22). Here $q = 1/\rho$.

as well as the system $\mathbf{EW}\{x, t\,;\, v, \sigma, \rho, w\}$ (5). We now calculate specific invariant solutions of the potential system $\mathbf{EW}\{x, t\,;\, v, \sigma, \rho, w\}$ (5) arising as reductions from a point symmetry that is a potential symmetry of the Euler system $\mathbf{E}\{x, t\,;\, v, \sigma, \rho\}$ (1).

We choose the constitutive relation $\sigma = K(\rho)$ given by

$$\sigma = K(\rho) = \frac{1}{2}\arctan\frac{1}{\rho} + \frac{1}{2}\frac{\rho}{\rho^2+1}, \tag{22}$$

(see Figure 2), and a linear body force $f(x) = x$. In this case, the potential Euler system $\mathbf{EW}\{x, t\,;\, v, \sigma, \rho, w\}$ (5) has the point symmetry

$$Y_4 = \frac{e^t}{\rho}\left[\frac{\partial}{\partial t} + (v+\rho w)\frac{\partial}{\partial x} + (x+\rho w)\frac{\partial}{\partial v} - \rho(\rho^2+1)\frac{\partial}{\partial \rho} - \rho(x-v)\frac{\partial}{\partial w}\right],$$

which is clearly a nonlocal symmetry of the Euler system $\mathbf{E}\{x, t\,;\, v, \sigma, \rho\}$ (1), since its x- and v-components depend on the potential variable w.

The physical dependent variables ρ, v, w, σ are found as functions of x and t. The velocity $v(x, t)$ solves the implicit equation:

$$v(x,t) = e^t\left(\frac{C(U)}{A(U)} - \frac{U}{A^2(U)}\right) \tag{23}$$

where $A(U)$ and $C(U)$ are given by

$$A(U) = \sqrt{U^2+\alpha^2}, \quad C(U) = \frac{1}{2}\frac{\alpha U + (U^2+\alpha^2)\left(\beta - \arctan\frac{U}{\alpha}\right)}{\alpha(U^2+\alpha^2)}, \tag{24}$$

with α, β constants of integration, and the similarity variable given by $U = e^t(x - v(x,t))$.

For every value of x and t, the solution $v(x, t)$ of (23) can be found numerically. After $v(x, t)$ (and thus the similarity variable U) is determined, the density and the mass coordinate are obtained from the formulas

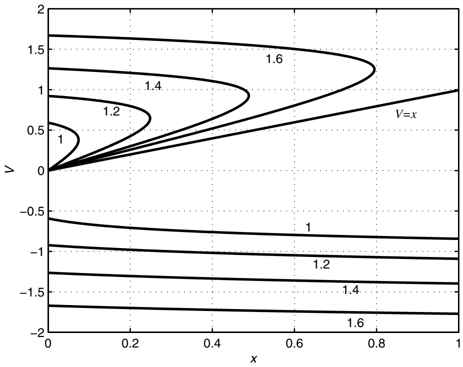

Fig. 3 Solution curves of the implicit equation (23) defining the material velocity $v(x,t)$ ($\alpha = 2$; $t = 1, 1.2, 1.4, 1.6$).

$$\rho(x,t) = \frac{\sqrt{U^2+\alpha^2}}{\sqrt{e^{2t}-U^2-\alpha^2}}, \quad w(x,t) = \frac{v(x,t)-x}{\rho(x,t)}.$$

We seek a solution describing a nonlinear deformation of an elastic slab $x_0 < x < L(t)$, attached at $x = x_0$ (i.e., subject to the boundary condition $v(x_0,t) = 0$). We use the boundary conditions

$$v(x_0,t) = 0, \quad \rho(x_0,t) = R(t), \quad w(x_0,t) = 0. \tag{25}$$

The latter boundary condition is due to the definition of the mass coordinate (potential variable): $w(x,t) = \int_{x_0}^{x} \rho(s,t)ds$. When substituted into equation (23), this boundary condition yields $\beta = 0, \quad x_0 = 0$.

The velocity $v(x,t)$ following from equation (23) turns out to be a three-valued function for $0 \le x < x^*(t)$, where $x^*(t)$ is a bifurcation point. Sample curves of $v(x,t)$ for $\alpha = 2$ and times $t = 1, 1.2, 1.4, 1.6$ are shown in Figure 3.

From the three possible values of $v(x,t)$ that arise from the implicit equation (23), only one branch is physical. Indeed, one may check that only the middle branch (the one closest to $v(x,t) = x$) yields a real-valued density function [7].

It is also important to note that the geometrical velocity of the bifurcation point $v^*(t) = dx^*(t)/dt$ is always greater than the physical velocity $v(x^*(t),t)$ at the bifurcation point. The expression for the total mass between $x = 0$ and the bifurcation point $x^*(t)$ (per unit area of the slab cross-section) is given by

$$w(x^*(t),t) = 1 - \alpha e^{-t}, \tag{26}$$

which is an increasing function of time, in agreement with the previous remark. The invariant solutions are defined for $0 < x < x^*(t)$. If the initial length of the slab $L(t_0)$ is chosen ($0 < L(t_0) < x^*(t_0)$), then the solution is regular for all times.

The family of invariant solutions presented in this section describes the nonlinear deviation of a trivial “homogeneous stretching" solution $v(x,t) = x$, $\rho(x,t) = e^{-t}$

of the Euler system $\mathbf{E}\{x, t\,;\, v, \sigma, \rho\}$ (1). A specific numerical example of such an invariant solution was constructed in [7].

6 Conservation Laws of Dynamical Nonlinear Elasticity

For a PDE system $\mathbf{R}\{x\,;\, u\}$ (3), one can consider the problem of finding local conservation laws of the form (4). The fluxes $\Phi^i[u]$ may depend on x, u and derivatives of u up to an arbitrary order. In practice, conservation laws are used for direct physical interpretation, analysis, and development of efficient numerical methods.

The direct method of finding conservation laws involves considering a linear combination of equations of a given PDE system (3) with a set of *multipliers* $\{\Lambda_\sigma\}$, which may depend on independent and dependent variables and their derivatives. A linear combination yields a conservation law (4) if and only if

$$\Lambda_\sigma[U]R^\sigma[U] \equiv D_i\Phi^i[U] \tag{27}$$

for some fluxes $\{\Phi^i[U]\}$ (here U denotes a vector of arbitrary functions of x). Then the conservation law $D_i\Phi^i[u] = 0$ holds on the solutions $U = u(x)$ of the system (3).

In the direct method, the determining equations that yield sets of multipliers $\{\Lambda_\sigma[U]\}$ are found from the known fact: an expression is a divergence expression if and only if it is annihilated by Euler operators with respect to all dependent variables [2, 13, 14]:

$$E_{U^k}\left(\Lambda_\sigma[U]R^\sigma[U]\right) = 0, \quad k = 1, \ldots, m. \tag{28}$$

Here $U = (U^1(x), \ldots, U^m(x))$ is a set of arbitrary functions, and E_{U^k} is the *Euler operator* with respect to U^k, given by

$$E_{U^k} = \frac{\partial}{\partial U^k} - D_i\frac{\partial}{\partial U^k_i} + \cdots + (-1)^j D_{i_1}\cdots D_{i_j}\frac{\partial}{\partial U^k_{i_1\cdots i_j}} + \cdots .$$

Symbols $U^k_{i_1\cdots i_j}$ denote partial derivatives $\frac{\partial^j U^k}{\partial x^{i_1}\ldots\partial x^{i_j}}$.

Equations (28) are linear determining equations for the multipliers $\{\Lambda_\sigma[U]\}$. In practice, to perform a computation, one chooses the maximal order of derivatives $q \geq 0$ in the dependence of multipliers $\Lambda_\sigma[U]$. When the multipliers are determined, one finds the corresponding set of fluxes of the conservation law (4) either by solving (27) directly, or using integral homotopy operators [13, 14].

Note that the direct method does not require the PDE system to have a variational formulation, and does not use any version of Noether's theorem.

Now our goal is to construct examples of nonlocal conservation laws of the Euler system $\mathbf{E}\{x, t\,;\, v, \sigma, \rho\}$ (1) of nonlinear elastodynamics, which arise as local conservation laws of a potential system of $\mathbf{E}\{x, t\,;\, v, \sigma, \rho\}$. The following important theorem holds [16].

Theorem 1. *Let* $\mathbf{S}\{x\,;\, u, v\}$ *given by*

$$S^{\mu}[u, v] \equiv S^{\mu}(x, u, v, \partial u, \partial v, \dots, \partial^k u, \partial^k v) = 0, \quad \mu = 1, \dots, M \tag{29}$$

be a potential system of a PDE system $\mathbf{R}\{x\,;\, u\}$ (3)*, where* $v = (v^1, \dots, v^l)$ *are nonlocal (potential) variables. A local conservation law*

$$\widetilde{\Lambda}_{\mu}[u, v] S^{\mu}[u, v] = D_i \Psi^i[u, v] = 0 \tag{30}$$

of the potential system $\mathbf{S}\{x\,;\, u, v\}$ (29) *yields a nonlocal conservation law of* $\mathbf{R}\{x\,;\, u\}$ (3) *if and only if the multipliers* $\widetilde{\Lambda}_{\mu}[u, v]$ *essentially depend on the nonlocal variable(s)* v.

We now seek local conservation laws of the potential system $\mathbf{EW}\{x, t\,;\, v, \sigma, \rho, w\}$ (5) that yield nonlocal conservation laws of the Euler system $\mathbf{E}\{x, t\,;\, v, \sigma, \rho\}$ (1), i.e., conservation laws of the potential system $\mathbf{EW}\{x, t\,;\, v, \sigma, \rho, w\}$ (5) arising from one or more of multipliers with an essential dependence on the potential variable w. One may write

$$\mathbf{EW}\{x, t\,;\, v, \sigma, \rho, w\} : \quad \begin{cases} w_x - \rho = 0, \\ w_t + \rho v = 0, \\ \dfrac{K'(\rho)}{\rho}\rho_x + f(x) - (v_t + v v_x) = 0. \end{cases} \tag{31}$$

For the conservation law multipliers, we use the ansatz

$$\widetilde{\Lambda}_{\mu} = \widetilde{\Lambda}_{\mu}(x, t, V, R, w, V_x, V_t), \quad \mu = 1, 2, 3,$$

and require that for arbitrary functions $V(x, t)$, $P(x, t)$, $W(x, t)$, one has

$$\begin{aligned} &\widetilde{\Lambda}_1(x, t, V, P, W, V_x, V_t)(W_x - R) + \widetilde{\Lambda}_2(x, t, V, P, W, V_x, V_t)(W_t + RV) \\ &+ \widetilde{\Lambda}_3(x, t, V, P, W, V_x, V_t)\left(\frac{K'(R)}{R}R_x + f(x) - (V_t + VV_x)\right) \equiv D_i \Psi^i[U, V]. \end{aligned} \tag{32}$$

Then on solutions $V = v$, $P = \rho$, $W = w$ of the PDE system $\mathbf{EW}\{x, t\,;\, v, \sigma, \rho, w\}$ (5), the expression (32) becomes a conservation law.

Subsequent application of the Euler operators with respect to V, P and W to the left-hand side of (32) yields determining equations for the multipliers $\widetilde{\Lambda}_{\mu}$, $\mu = 1, 2, 3$. From the determining equations, it follows that one or more of the multipliers essentially depend on W, and thus a nonlocal conservation law of the Euler system $\mathbf{E}\{x, t\,;\, v, \sigma, \rho\}$ (1) arises in the following cases.

Case 1: $f(x) = f_1 = \text{const}$, $K(\rho) = \frac{\rho^{1/3}}{(A\rho + B)^{1/3}} + C$, $A, B \neq 0$. In this case, the multipliers are given by

$$
\begin{aligned}
\Lambda_1 &= V - f_1 t + V_x \left(f_1 \frac{t^2}{2} - x - \frac{3AW}{B} \right), \\
\Lambda_2 &= V^2 + xV_t + f_1 \left[\frac{t^2}{2}(1 - V_t) - x - tV \right] + \frac{3AW}{B}(V_t - f_1) + \frac{P^{-2/3}}{(AP+B)^{1/3}}, \\
\Lambda_3 &= P \left(f_1 \frac{t^2}{2} - x - \frac{3AW}{B} \right) - 2W.
\end{aligned}
\tag{33}
$$

Case 2: $f(x) = f_0 x + f_1$, $K(\rho) = A\rho^{1/3} + B$, $A, B, f_0, f_1 = \text{const}$. The multipliers are given by

$$
\begin{aligned}
\Lambda_1 &= f_0(V - xV_x) - f_1 V_x, \\
\Lambda_2 &= -f_0^2 x^2 + f_0(x(V_t - 2f_1) + V^2) + f_1(V_t - f_1) + Af_0 P^{-2/3}, \\
\Lambda_3 &= -f_0(xP + 2W) - f_1 P.
\end{aligned}
\tag{34}
$$

Case 3: $f(x) = f_1$, $K(P) = A\rho^{1/3} + B$. The multipliers are given by

$$
\begin{aligned}
\Lambda_1 &= -f_1 t + V - V_x \left(x - f_1 \frac{t^2}{2} \right), \\
\Lambda_2 &= f_1^2 - f_1 \left(x + tV + \frac{t^2}{2} V_t \right) + V^2 + xV_t + AP^{-2/3}, \\
\Lambda_3 &= -\left(x - f_1 \frac{t^2}{2} \right) P - 2W.
\end{aligned}
\tag{35}
$$

Case 4: $f(x) = f_0 x + f_1$, $K(\rho) = \rho_0/\rho$, $\rho_0 = \text{const}$. This case corresponds to linear elasticity with linear loading. Here one finds that the potential system $\mathbf{EW}\{x, t\,; v, \sigma, \rho, w\}$ (5) has an infinite number of conservation laws corresponding to nonlocal conservation laws of the Euler equations $\mathbf{E}\{x, t\,; v, \sigma, \rho\}$. This reflects the fact that for linear elasticity with linear loading, the system $\mathbf{EW}\{x, t\,; v, \sigma, \rho, w\}$ (5) can be linearized by a point transformation [17]. Indeed, this transformation is the interchange of dependent and independent variables that transforms the system $\mathbf{EW}\{x, t\,; v, \sigma, \rho, w\}$ (5) to the Lagrange system $\mathbf{L}\{y, s\,; v, \sigma, q, x\}$ (2) (see Section 3).

7 Conclusions

In this paper, we presented the complete set of dynamic nonlinear elasticity equations in Lagrangian and Eulerian formulations, as well as in several other equivalent formulations. The corresponding nonlocally related systems were used for the classification of nonlocal symmetries and construction of examples of invariant solutions of the Euler system $\mathbf{E}\{x, t\,; v, \sigma, \rho\}$ (1). Moreover, we demonstrated how nonlocal conservation laws can be obtained for the Euler system through consideration of local conservation laws of its potential system.

Future work will include flux computation, interpretation and applications of the nonlocal conservation laws of the Euler system $\mathbf{E}\{x, t\,; v, \sigma, \rho\}$ (1) obtained in this paper, and also the study of conservation laws of two-dimensional models of nonlinear elasticity.

References

1. Bluman, G.W. and Anco, S.C., *Symmetry and Integration Methods for Differential Equations*. Springer, New York, 2002.
2. Olver, P.J., *Application of Lie Groups to Differential Equations*. Springer, New York, 1993.
3. Ovsiannikov, L.V., *Group Analysis of Differential Equations*. Academic Press, New York, 1982.
4. Bluman, G.W. and Kumei, S., *Symmetries and Differential Equations*. Springer, New York, 1989.
5. Dorodnitsyn, V. and Winternitz, P., Lie point symmetry preserving discretizations for variable coefficient Korteweg–de Vries equations. Modern group analysis. *Nonlin. Dynam.* **22**, 2000, 49–59.
6. Maugin, G.A., *Material Inhomogeneities in Elasticity*. Cambridge University Press, 1993.
7. Bluman, G., Cheviakov, A. and Ganghoffer, J.F., Nonlocally related PDE systems for one-dimensional nonlinear elastoydnamics. *Int. J. Engng Math.*, 2008, doi: 10.1007/s10665-008-9221-7.
8. Horgan, C.O. and Murphy, J.H., Lie group analysis and plane strain bending of cylindrical sectors for compressible nonlinearly elastic materials. *IMA J. Appl. Math.* **70**, 2005, 80–91.
9. Horgan, C.O. and Murphy, J.H., A Lie group analysis of the axisymmetric equations of finite elastostatics for compressible materials. *Math. Mech. Sol.* **10**, 2005, 311–333.
10. Budiansky, B. and Rice, J.R., Conservation laws and energy release rates. *J. Appl. Mech.* **40**, 1968, 201–203.
11. Hatfield, G.A. and Olver P.J., Canonical forms and conservation laws in linear elastostatics. *Arch. Mech.* **50**, 1998, 389–404.
12. Bluman, G., Cheviakov, A.F. and Ivanova, N.M., Framework for nonlocally related PDE systems and nonlocal symmetries: extension, simplification, and examples. *J. Math. Phys.* **47**, 2006, 113505.
13. Anco, S. and Bluman, G., Direct construction method for conservation laws of partial differential equations. Part I: Examples of conservation law classfications. *Eur. J. Appl. Math.* **13**, 2002, 545–566.
14. Anco, S. and Bluman, G., Direct construction method for conservation laws of partial differential equations. Part II: General treatment. *Eur. J. Appl. Math.* **13**, 2002, 567–585.
15. Cheviakov, A.F., An extended procedure for finding exact solutions of partial differential equations arising from potential symmetries. Applications to gas dynamics. *J. Math. Phys.* **49**, 2008, 083502.
16. Kunzinger, M. and Popovych, R.O., *Potential Conservation Laws*. arXiv:0803.1156v2 [math-ph], 2008.
17. Anco, S., Bluman, G. and Wolf, T., Invertible mappings of nonlinear pdes through admitted conservation laws, *Acta Appl. Math.* **101**, 2008, 21–38.