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An extended procedure for finding exact solutions of partial differential equations arising from potential symmetries. Applications to gas dynamics

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Lie point symmetries and nonlocal symmetries of partial differential equation (PDE) systems are widely used for construction of exact invariant solutions. In this paper we describe an extended algorithmic procedure that, for a given nonlocal (potential) symmetry, can yield additional exact solutions, which cannot be found using the usual algorithm. In particular, such additional solutions are exact solutions of the given PDE system, but are not invariant solutions of the corresponding potential system. As an example, we consider a tree of nonlocally related PDE systems for Lagrange planar gas dynamics equations and classify its nonlocal symmetries for an ideal polytropic gas. For two different nonlocal symmetries of the Lagrange system, we demonstrate that the extended method yields wider classes of exact solutions than the usual method. © *2008 American Institute of Physics.*
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I. INTRODUCTION

An admitted symmetry of a system of partial differential equations (PDEs) is any transformation of its solution manifold into itself. In particular, Lie groups of local (point, contact, and higher order) symmetries and discrete symmetries can be found using Lie algorithm.[1,2] One of the most important applications of such continuous symmetry groups is the construction of exact symmetry-invariant solutions of nonlinear PDE systems. Symmetry transformations are also often used to generate new exact solutions from known ones.

It has been demonstrated in many papers that Lie groups of local symmetries do not include all calculable (as well as useful) symmetries of a given PDE system. Indeed, many PDE systems admit *nonlocal symmetries* that arise as local symmetries of nonlocally related PDE systems.

For any given PDE system, one can systematically construct a set (a *tree*) of PDE systems which are nonlocally related to it.[3,4] In particular, a *potential system* is obtained by augmenting the given system with potential equations following from an admitted conservation law. Most importantly, solution sets of nonlocally related systems are equivalent: each solution of a potential system projects onto a solution of the given PDE system, and conversely, each solution of the given PDE yields a solution of the potential system. (Consequently, the solution of any boundary value problem posed for the given PDE system is embedded in the solution of a boundary value problem posed for the potential system, and the converse also holds.) Nonlocally related *subsystems* can be obtained by exclusion of dependent variables through differential consequences. Due to the nonlocal relation and the equivalence of solution sets, any general method of analysis (conservation law, qualitative, perturbation, numerical, etc.), and particularly symmetry methods, when applied to one of nonlocally related PDE systems, can yield new results for the given system.

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Trees of nonlocally related systems have recently been constructed and studied for various general equations of mathematical physics (e.g., Refs. 3–10), yielded multiple new results, including new admitted symmetries, conservation laws, and linearizations. Nonlocal symmetries and resulting exact invariant solutions were found for many PDE systems, including such important physical models a nonlinear wave and diffusion equations, equations of gas and plasma dynamics, nonlinear elasticity, Maxwell's equations, and other models (e.g., Refs. 3–5 and 9–14).

This paper is devoted to the question of finding exact solutions of a given PDE system using its admitted nonlocal (potential) symmetry. The "usual" approach here is to simply construct invariant solutions of the corresponding potential system, and then project them onto the space of variables of the given system. In this paper, we provide an extended algorithm that is capable of yielding larger classes of exact solutions, in particular, solutions that cannot be obtained using the usual approach.

The extended algorithm is straightforward in its application and does not result in a serious increase in the number of computations. We argue that the extended algorithm should be used as a standard procedure for finding solutions arising from nonlocal symmetries.

The rest of the paper is organized as follows. In Sec. II, we review the procedure of construction of nonlocally related PDE systems. As an example, we construct an extended tree of nonlocally related systems for polytropic planar gas dynamics (PGD) equations. Here we incorporate and systematically extend the work results of the previous papers.[3,4,9,13] In Sec. III, we review the concept of nonlocal symmetries. We present a comparative symmetry classification of several nonlocally related systems of polytropic PGD equations and isolate nonlocal symmetries. In Sec. IV, we review the usual procedure of finding exact solutions invariant with respect to an admitted nonlocal symmetry, discuss generalization ideas, and present the extended algorithm of finding exact solutions. Finally, in Sec. V, we use the extended algorithm to construct exact solutions of the Lagrange polytropic PGD system and demonstrate that it yields additional families of exact solutions which do not arise as invariant solution of the original or potential PDE system.

In this paper, for simplicity of presentation, we consider only the PDE systems with two independent variables (x,t). However the proposed method naturally carries over to the multidimensional case (three or more independent variables).

The symbolic software package "GEM" for "MAPLE"[15] was used for symmetry and conservation law computations.

II. CONSTRUCTION OF PDE SYSTEMS NONLOCALLY RELATED TO A GIVEN ONE

A. Conservation laws

Let $\mathbf{R}\{x,t;\mathbf{u}\}$ be PDE system of m equations with two independent variables (x,t) and n dependent variables $\mathbf{u}=(u^1(x,t),\ldots,u^n(x,t))$,

$$R_i\{x,t;\mathbf{u}\}=0, \quad i=1,\ldots,m. \tag{2.1}$$

Its local admitted conservation laws are given by divergence expressions

$$D_t\Phi(x,t,\mathbf{u},\partial\mathbf{u},\ldots,\partial^r\mathbf{u})+D_x\Psi(x,t,\mathbf{u},\partial\mathbf{u},\ldots,\partial^r\mathbf{u})=0. \tag{2.2}$$

Here the total derivative operators are given by

$$D_i=\frac{\partial}{\partial x_i}+u_i\frac{\partial}{\partial u}+u_{ij_1}\frac{\partial}{\partial u_{j_1}}+\cdots+u_{ij_1j_2\cdots j_{k-1}}\frac{\partial}{\partial u_{j_1j_2\cdots j_r}}, \quad i,j_l=1,2,$$

with $x=(x_1,x_2)$, $x_1=t$, $x_2=x$, $D_1=D_t$, and $D_2=D_x$. (Here and below, summation in repeated indices is assumed.) Partial derivatives are denoted by $u_i^k=\partial u^k(x)/\partial x^i$; $\partial\mathbf{u}$ is the vector of first partial derivatives; $\partial^p\mathbf{u}$ is the vector of partial derivatives of order p.

For a nondegenerate (Cauchy–Kovalevskaya type) PDE system, each of its admitted conservation laws can be expressed as a linear combination of the equations of the system only,[2]

$$\Lambda^i(x,t;\mathbf{u},\partial\mathbf{u},\ldots,\partial^l\mathbf{u})R_i\{x,t;\mathbf{u}\} = D_t\Phi + D_x\Psi = 0. \tag{2.3}$$

with some multipliers $\{\Lambda^k\}_{k=1}^m$. Sets of multipliers that lead admitted conservation laws of $\mathbf{R}\{x,t;\mathbf{u}\}$ can be systematically found, for any prescribed dependence of multipliers $\Lambda^k=\Lambda^k(x,t;\mathbf{u},\partial\mathbf{u},\ldots)$, from corresponding determining equations involving Euler operators.[16,17] When multipliers are determined, the density Φ and the flux Ψ are reconstructed using one of the available methods.[16–18] (In multidimensions, the above-described method yields all admitted divergence-type conservation laws div $\mathbf{\Psi}=0$ of a given PDE system.)

Software packages such as GEM for MAPLE (Ref. 15) and "CRACK/CONLAW" for "REDUCE" are used for automated computation of multipliers and fluxes/densities of admitted conservation laws.

B. Potential systems

A conservation law (2.2) can be used to introduce a *potential variable* $v=v(x,t)$, satisfying a pair of potential equations,

$$\mathcal{P}:\begin{cases} v_x = \Phi(x,t,\mathbf{u},\partial\mathbf{u},\ldots,\partial^r\mathbf{u}) \\ v_t = -\Psi(x,t,\mathbf{u},\partial\mathbf{u},\ldots,\partial^r\mathbf{u}). \end{cases} \tag{2.4}$$

A potential variable v is normally a *nonlocal variable*, functionally independent of $x,t,\mathbf{u}$ and derivatives. The corresponding *potential system* is given by the union of $\mathbf{R}\{x,t;\mathbf{u}\}$ (2.1) and the potential equations,

$$\mathbf{S}\{x,t;\mathbf{u},v\} = \mathbf{R}\{x,t;\mathbf{u}\} \cup \mathcal{P}. \tag{2.5}$$

By construction, the potential system $\mathbf{S}\{x,t;\mathbf{u},v\}$ (2.5) has a solution set equivalent to that of the given PDE system $\mathbf{R}\{x,t;\mathbf{u}\}$. Indeed, if $\mathbf{u}=\mathbf{\Theta}(x,t)$ solves $\mathbf{R}\{x,t;\mathbf{u}\}$, then due to satisfaction of the integrability condition $v_{xt}=v_{tx}$, there exists a corresponding solution $v=\Xi(x,t)$ of the potential system $\mathbf{S}\{x,t;\mathbf{u},v\}$, defined uniquely up to a constant. Conversely, if $(u,v)=(\mathbf{\Theta}(x,t),\Xi(x,t))$ solves the potential system, then by projection, $u=\mathbf{\Theta}(x,t)$ solves $\mathbf{R}\{x,t;\mathbf{u}\}$.

Suppose now that for the given system $\mathbf{R}\{x,t;\mathbf{u}\}$, $N>1$ linearly independent conservation laws are known, with corresponding potential variables v^j defined by potential equations $\mathcal{P}^j$, $j=1,\ldots,N$. One can consider N singlet potential systems $\mathbf{S}^{(1)}\{x,t;u,v^j\}=\mathbf{R}\{x,t;\mathbf{u}\}\cup\mathcal{P}^j$ involving single potentials, $n(n-1)/2$ couplet potential systems $\mathbf{S}^{(2)}\{x,t;u,v^{j_1},v^{j_2}\}=\mathbf{R}\{x,t;\mathbf{u}\}\cup\mathcal{P}^{j_1}\cup\mathcal{P}^{j_2}$ involving pairs of potentials, …, and one n-plet $\mathbf{S}^{(N)}\{x,t;u,v^1,\ldots,v^N\}=\mathbf{R}\{x,t;\mathbf{u}\}\cup\mathcal{P}^1\cup\cdots\cup\mathcal{P}^N$ involving N potentials.

Definition 1. Suppose N linearly independent local conservation laws are known for a given PDE system $\mathbf{R}\{x,t;\mathbf{u}\}$. In terms of the resulting potential variables $v^1,\ldots,v^N$, the set of all corresponding 2^N-1 potential systems is called a *combination potential system* $P_{v^1\ldots v^N}$.

For details and examples, see also Refs. 4 and 5.

C. Subsystems

Another important way of finding PDE systems that are nonlocally related and equivalent to a given PDE system is the construction of appropriate subsystems. Suppose $\mathbf{R}\{x,t;\mathbf{u}\}$ has m dependent variables $u=(u^1,\ldots,u^m)$. A subsystem is a PDE system obtained from $\mathbf{R}\{x,t;\mathbf{u}\}$ by excluding one or more of its dependent variables, with properties that (1) any solution of the subsystem yields a solution of $\mathbf{R}\{x,t;\mathbf{u}\}$, and (2) that the solutions of the subsystem yield all solutions of $\mathbf{R}\{x,t;\mathbf{u}\}$. Hence a subsystem is equivalent to $\mathbf{R}\{x,t;\mathbf{u}\}$. Subsystems can arise directly through elimination of given dependent variables of $\mathbf{R}\{x,t;\mathbf{u}\}$, as well as indirectly through elimination of dependent variables following a point transformation that involves an interchange of one or more dependent and independent variables of $\mathbf{R}\{x,t;\mathbf{u}\}$.

In practice, one is usually interested in *nonlocally related subsystems*. Such subsystems commonly arise in the following situations.

- Direct exclusion from $\mathbf{R}\{x,t;\mathbf{u}\}$ of one or more of its dependent variables u^s that are present in equations only in terms of derivatives.
- Exclusion of dependent variable(s) after interchange of one or more dependent and independent variables of $\mathbf{R}\{x,t;\mathbf{u}\}$.

We denote a subsystem obtained from $\mathbf{R}\{x,t;u^1,\ldots,u^m\}$ by excluding u^1 as $\underline{\mathbf{R}}\{x,t;u^2,\ldots,u^m\}$.

D. Construction of an extended tree of nonlocally related PDE systems

A practically efficient procedure of construction of an extended tree of nonlocally related PDE systems for a given system $\mathbf{R}\{x,t;\mathbf{u}\}$ was first suggested in Ref. 4 and is outlined below.

(1) *Construction of conservation laws.* Find a set of linearly independent and inequivalent local conservation laws admitted by $\mathbf{R}\{x,t;\mathbf{u}\}$. Let N be the number of such conservation laws that are found.

(2) *Construction of potential systems.* Use the N known conservation laws to introduce N potential variables v^j. Construct the corresponding combination potential system $P_{v^1\ldots v^N}$ which contains 2^N-1 potential systems. Together with the given system $\mathbf{R}\{x,t;\mathbf{u}\}$, this yields a tree $\mathcal{T}_1$ with up to 2^N nonlocally related systems.

(3) *Additional conservation laws.* In the tree $\mathcal{T}_1$, consider the N-plet potential system $\mathbf{S}^{(N)}\{x,t;u,v^1,\ldots,v^N\}$. For this N-plet, seek its linearly independent conservation laws. Eliminate conservation laws that are linearly dependent on local conservation laws of $\mathbf{R}\{x,t;\mathbf{u}\}$. Let the number of newly obtained linearly independent conservation laws of $\mathbf{S}^{(N)}\{x,t;u,v^1,\ldots,v^N\}$ be N_1. Introduce corresponding potential variables $v^j, j=N+1,\ldots,N+N_1$. (By construction, the full set of potentials $\{v^1,\ldots,v^{N+N_1}\}$ is linearly independent.)

(4) *Tree extension.* Use the $N+N_1$ potentials $\{v^j\}$ to construct the corresponding combination potential system $P_{v^1\ldots v^{N+N_1}}$. Together with the given system $\mathbf{R}\{x,t;\mathbf{u}\}$, this yields an extended tree $\mathcal{T}_2$.

(5) *Continuation.* Repeat steps 3 and 4 for the tree $\mathcal{T}_2$, until no further linearly independent conservation laws are found for any nonlocally related potential system. This yields a possibly larger extended tree $\mathcal{T}_3$.

(6) *Construction of subsystems.* For all systems in the tree $\mathcal{T}_3$, exclude where possible, one by one, dependent variables, to generate subsystems of the systems in the tree $\mathcal{T}_3$. Eliminate locally related subsystems. In addition, in the same manner, generate nonlocally related subsystems obtained after an interchange of one or more independent and independent variables. This yields a possibly larger extended tree of nonlocally related systems denoted by $\mathcal{T}_4$.

If the given PDE system $\mathbf{R}\{x,t;\mathbf{u}\}$ includes arbitrary constitutive function(s), one may be able to still further extend the above procedure: in steps 1, 3, and 6, isolate cases for which additional conservation laws and/or additional nonlocally related subsystems arise. Trees for particular forms of the constitutive functions could be significantly different, although sharing a common part that holds for arbitrary constitutive function(s).

Trees of nonlocally related PDE systems have been obtained in literature for nonlinear telegraph equations,[4] nonlinear wave equation,[5] nonlinear diffusion-convection equation,[7] and equations of nonlinear elasticity.[8] See also the related work of Bluman and Doran-Wu[6] on the nonlinear diffusion equation.

Remark 1. For PDE systems with $N\geq 3$ independent variables, potential systems arise in a way similar to the two-dimensional case. Divergence-type and lower-degree conservation laws can be used to introduce potential variables (see Refs. 1, 4, 10, and 19). Ideas related to invariant solutions presented below can be directly applied to the multidimensional case.

TABLE I. Local conservation laws and resulting potential equations for the Lagrange PGD system (2.6), arising from multipliers that are functions of independent variables.

Multipliers $(\Lambda_1,\Lambda_2,\Lambda_3)$	Conservation law	Potential variable	Potential equations
(1,0,0)	$D_s(q)-D_y(v)=0$	w^1	$w^1_y=q,\ w^1_s=v$
(0,1,0)	$D_s(v)+D_y(p)=0$	w^2	$w^2_y=v,\ w^2_s=-p$
$(y,s,0)$	$D_s(sv+yq)+D_y(sp-yv)=0$	w^3	$w^3_y=vs+qy,\ w^3_s=-sp+vy$
$(S_Q(P,Q),0,S_P(P,Q))$	$D_s(S(p,q))=0$	w^4	$w^4_y=S(p,q),\ w^4_s=0$
$(K_Q(P,Q),V,K_P(P,Q))$	$D_s\left(\frac{v^2}{2}+K(p,q)\right)+D_y(pv)=0$	w^5	$w^5_y=v^2/2+K(p,q),\ w^5_s=-pv$
$K_q(p,q)=B(p,q)K_p(p,q)-p$			

E. An extended tree of nonlocally related PDE systems for PGD equations

Consider the Lagrange PDE system $\mathbf{L}\{y,s;v,p,q\}$ given by

$$q_s - v_y = 0,$$

$$v_s + p_y = 0, \tag{2.6}$$

$$p_s + B(p,q)v_y = 0.$$

In (2.6), v is the gas velocity, $q=1/\rho$, ρ is the gas density, p is the gas pressure, $B(p,q)=S_q/S_p$ is the constitutive function, and S is entropy. The independent variables are time s and the Lagrange mass coordinate $y=\int_{x_0}^{x}\rho(\xi,t)d\xi$, where x is the usual Eulerian spatial coordinate of this one-dimensional problem.

First we note that system (2.6) admits the group of equivalence transformations,

$$\tilde{s}=a_1 s + a_4, \quad \tilde{y}=a_2 y + a_5, \quad \tilde{v}=a_3 v + a_6,$$
$$\tilde{p}=\frac{a_2 a_3}{a_1}p + a_7, \quad \tilde{q}=\frac{a_1 a_3}{a_2}q + a_8, \quad \tilde{B}(\tilde{p},\tilde{q})=\frac{a_2^2}{a_1^2}B(p,q) \tag{2.7}$$

for arbitrary constants $a_1,\ldots,a_8$ with $a_1a_2a_3\neq 0$. All further analysis is done modulo these equivalence transformations.

Using the algorithm described in Sec. II A for finding local conservation laws, one finds that assuming $\Lambda_i=\Lambda_i(y,s,V,P,Q)$, $i=1,2,3$, yields five independent conservation laws of the Lagrange PGD system (2.6).[4] These five conservation laws are listed in Table I and correspond to, respectively, conservation of mass, conservation of momentum, center of mass theorem, conservation of entropy, and conservation of energy.

The five singlet potential systems including single potential variables $w^1,\ldots,w^5$ are given by

$$\mathbf{LW}^1\{y,s;v,p,q,w^1\}:\begin{cases} w^1_y = q \\ w^1_s = v \\ v_s + p_y = 0 \\ p_s + B(p,q)v_y = 0, \end{cases} \tag{2.8}$$

$$\mathbf{LW}^2\{y,s;v,p,q,w^2\}:\begin{cases} q_s - v_y = 0 \\ w^2_y = v \\ w^2_s = -p \\ p_s + B(p,q)v_y = 0, \end{cases} \tag{2.9}$$

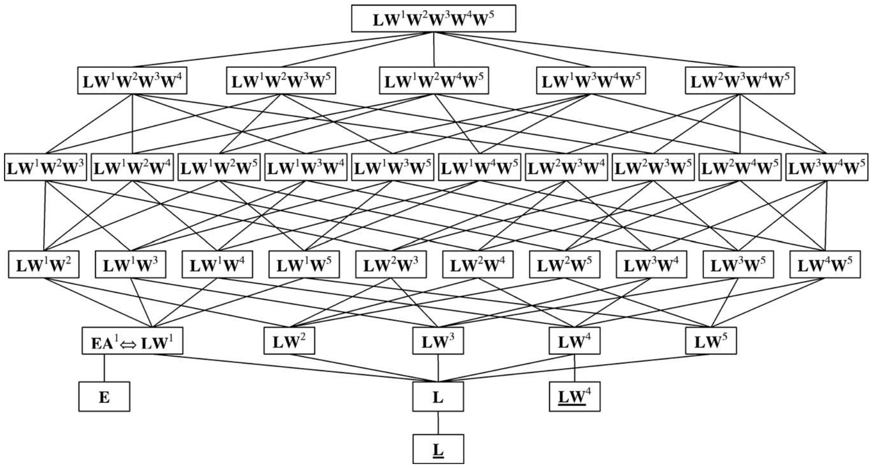

FIG. 1. An extended tree of nonlocally related systems for the PGD equations for an arbitrary constitutive function $B(p,q)$.

$$\mathbf{LW}^3\{y,s;v,p,q,w^3\}:\begin{cases} w^3_y = sv + yq \\ w^3_s = -sp + yv \\ v_s + p_y = 0 \\ p_s + B(p,q)v_y = 0, \end{cases} \tag{2.10}$$

$$\mathbf{LW}^4\{y,s;v,p,q,w^4\}:\begin{cases} w^4_y = S(p,q) \\ w^4_s = 0 \\ v_s + p_y = 0 \\ p_s + B(p,q)v_y = 0, \end{cases} \tag{2.11}$$

$$\mathbf{LW}^5\{y,s;v,p,q,w^5\}:\begin{cases} w^5_y = \dfrac{v^2}{2} + K(p,q) \\ w^5_s = -pv \\ v_s + p_y = 0 \\ p_s + B(p,q)v_y = 0. \end{cases} \tag{2.12}$$

Considering couplets, triplets, quadruplets, and the 5-plet, one obtains combination potential system $\mathbb{P}_{w^1\ldots w^5}$ containing the total of 31 nonlocally related potential systems, which are shown in Fig. 1.

Through the direct exclusion of dependent variables by differential consequences, one obtains a nonlocally related subsystem,

$$\underline{\mathbf{L}}\{y,s;p,q\}:\begin{cases} q_{ss} + p_{yy} = 0 \\ p_s + B(p,q)q_s = 0 \end{cases} \tag{2.13}$$

of the Lagrange system $\mathbf{L}\{y,s;v,p,q\}$ (2.6). Another nonlocally related subsystem follows from the singlet potential system $\mathbf{LW}^4\{y,s;v,p,q,w^4\}$ after excluding the dependent variable v, and is given by

$$\underline{\mathbf{LW}}^4\{y,s;p,q,w^4\}:\begin{cases} q_{ss}+p_{yy}=0 \\ w^4_y=S(p,q) \\ w^4_s=0 \\ p_s+B(p,q)q_s=0 \\ S_q(p,q)=B(p,q)S_p(p,q). \end{cases} \tag{2.14}$$

Turning to exclusions after interchanges of variables, following Ref. 3, we consider a local (point) coordinate transformation of the Lagrange potential system $\mathbf{LW}^1\{y,s;v,p,q,w^1\}$ (2.8) with $s=t$ and $w^1=x$ treated as independent variables and $y=\alpha^1,v,p,\rho=1/q$ as dependent variables (an interchange of w^1 and y variables). Without loss of generality, $\rho=1/q\neq 0$. We obtain the invertibly equivalent system,

$$\mathbf{EA}^1\{x,t;v,p,\rho,\alpha^1\}:\begin{cases} \alpha^1_x-\rho=0 \\ \alpha^1_t+\rho v=0 \\ \rho(v_t+vv_x)+p_x=0 \\ \rho(p_t+vp_x)+B(p,1/\rho)v_x=0. \end{cases}$$

Its nonlocally related subsystem is the well-known *Euler system* of gas dynamics equations,

$$\mathbf{E}\{x,t;v,p,\rho\}:\begin{cases} \rho_t+(\rho v)_x=0 \\ \rho(v_t+vv_x)+p_x=0 \\ \rho(p_t+vp_x)+B(p,1/\rho)v_x=0. \end{cases} \tag{2.15}$$

The tree of nonlocally related systems displayed in Fig. 1 contains the total of 35 PDE systems which are equivalent descriptions of gas dynamics equations.

Thus the two well-known (Eulerian and Lagrangian) related formulations of PGD, as well as many other formulations, arise naturally in the general abstract mathematical framework of nonlocally related PDE systems.

III. NONLOCAL SYMMETRIES OF PDEs

A. Point and local symmetries

The application of Lie method to a given PDE system $\mathbf{R}\{x,t;\mathbf{u}\}$ yields Lie groups of admitted point symmetry transformations, locally given by

$$x=x^i+\epsilon\xi(x,t,\mathbf{u})+O(\epsilon^2),$$

$$t=t^i+\varepsilon\tau(x,t,\mathbf{u})+O(\epsilon^2), \tag{3.1}$$

$$(u')^j=u^j+\varepsilon\eta^j(x,t,\mathbf{u})+O(\epsilon^2),\quad j=1,\ldots,n.$$

Here ε is a group parameter. The corresponding Lie algebra infinitesimal symmetry generators is given by vector fields,

$$\mathrm{X}=\xi(x,t,\mathbf{u})\frac{\partial}{\partial x}+\tau(x,t,\mathbf{u})\frac{\partial}{\partial t}+\eta^j(x,t,\mathbf{u})\frac{\partial}{\partial u^j}. \tag{3.2}$$

The global form of symmetry transformations corresponding to a one-parameter group (3.1) is given by

$$x'=f^1(x,t,\mathbf{u};\varepsilon)=e^{\varepsilon \mathrm{X}}x,\quad t'=f^2(x,t,\mathbf{u};\varepsilon)=e^{\varepsilon \mathrm{X}}t,\quad (u')^j=g^j(x,t,\mathbf{u};\varepsilon)=e^{\varepsilon \mathrm{X}}u^j, \tag{3.3}$$

$j=1,\ldots,n$. Symmetry components ξ,τ,η^j are solutions of a linear overdetermined system of determining equations, where the dependent variables $\mathbf{u}$ and their derivatives are treated as arbitrary functions.[1] In the same manner, one may look for other types of local symmetries: *contact* and *higher-order symmetries* of PDE systems. There symmetry components ξ, τ, η^j depend on derivatives of the dependent variable(s).[1]

B. Nonlocal symmetries

One can isolate three main types of nonlocal symmetries that can be sought for a given PDE system $\mathbf{R}\{x,t;\mathbf{u}\}$: (1) nonlocal symmetries (potential symmetries) that arise as point symmetries of potential systems of $\mathbf{R}\{x,t;\mathbf{u}\}$; (2) nonlocal symmetries arising from nonlocally related subsystems of $\mathbf{R}\{x,t;\mathbf{u}\}$; (3) nonlocal symmetries arising from nonlocally related subsystems of potential systems of $\mathbf{R}\{x,t;\mathbf{u}\}$.

1. Potential symmetries

Suppose a system of PDEs $\mathbf{R}\{x,t;\mathbf{u}\}$ has a potential system (k-plet) $\mathbf{S}\{x,t;\mathbf{u},\mathbf{v}\}$ that admits a one-parameter (ε) Lie group of point transformations,

$$\begin{aligned} x' &= x + \varepsilon \xi_S(x,t,\mathbf{u},\mathbf{v}) + O(\epsilon^2), \\ t' &= t + \varepsilon \tau_S(x,t,\mathbf{u},\mathbf{v}) + O(\epsilon^2), \\ (u')^j &= u^j + \varepsilon \eta_S^j(x,t,\mathbf{u},\mathbf{v}) + O(\epsilon^2), \quad j=1,\ldots,n, \\ (v')^i &= v^i + \varepsilon \zeta_S^i(x,t,\mathbf{u},\mathbf{v}) + O(\epsilon^2), \quad i=1,\ldots,k, \end{aligned} \tag{3.4}$$

with corresponding infinitesimal generator,

$$\mathrm{X} = \xi_S(x,t,\mathbf{u},\mathbf{v})\frac{\partial}{\partial x} + \tau_S(x,t,\mathbf{u},\mathbf{v})\frac{\partial}{\partial t} + \eta_S^j(x,t,\mathbf{u},\mathbf{v})\frac{\partial}{\partial u^j} + \zeta_S^i(x,t,\mathbf{u},\mathbf{v})\frac{\partial}{\partial v^i}. \tag{3.5}$$

The group of transformations (3.4) map any solution of $\mathbf{S}\{x,t;\mathbf{u},\mathbf{v}\}$ to a solution of $\mathbf{S}\{x,t;\mathbf{u},\mathbf{v}\}$, and hence through projection, induces a mapping of any solution of $\mathbf{R}\{x,t;\mathbf{u}\}$ to a solution of $\mathbf{R}\{x,t;\mathbf{u}\}$. Thus (3.4) yields a symmetry group of $\mathbf{R}\{x,t;\mathbf{u}\}$.

If the infinitesimals $(\xi_S, \tau_S, \eta_S^j)$ explicitly depend on the nonlocal variables $\mathbf{v}$, i.e.,

$$\sum_i \left(\frac{\partial \xi_S}{\partial v^i}\right)^2 + \sum_i \left(\frac{\partial \tau_S}{\partial v^i}\right)^2 + \sum_{i,j} \left(\frac{\partial \eta_S^j}{\partial v^i}\right)^2 > 0, \tag{3.6}$$

then the transformation (3.4) defines a nonlocal (potential) symmetry admitted by $\mathbf{R}\{x,t;u\}$. Otherwise, (3.4) corresponds to a point symmetry admitted by $\mathbf{R}\{x,t;u\}$.

2. Nonlocal symmetries arising from nonlocally related subsystems

Now suppose a system of PDEs $\mathbf{R}\{x,t;\mathbf{u}\}=\mathbf{R}\{x,t;u^1,\ldots,u^m\}$ has a nonlocally related subsystem $\underline{\mathbf{R}}\{x,t;u^{s+1},\ldots,u^m\}$ obtained by excluding (without loss of generality) dependent variable(s) $u^1,\ldots,u^s$ $(1 \le s \le m-1)$. [Note that one may also consider subsystems arising from exclusions of dependent variables after transformations involving interchanges of dependent/independent variables or after other point transformations. The Euler system (2.15) of PGD equations serves as an example.]

Since the systems $\mathbf{R}\{x,t;\mathbf{u}\}$ and $\underline{\mathbf{R}}\{x,t;u^{s+1},\ldots,u^m\}$ are nonlocally related, their sets of admitted local symmetries may differ. In order to isolate nonlocal symmetries arising from a subsystem, one must find all point symmetries admitted by the subsystem and compare them to admitted local symmetries of $\mathbf{R}\{x,t;\mathbf{u}\}$.

TABLE II. Point symmetries of PGD systems $\mathbf{E}\{x,t;v,p,\rho\}$, $\mathbf{L}\{y,s;v,p,q\}$ and $\underline{\mathbf{L}}\{y,s;p,q\}$ in the polytropic case.

	Admitted point symmetries		
γ	$\mathbf{E}\{x,t;v,p,\rho\}$	$\mathbf{L}\{y,s;v,p,q\}$	$\underline{\mathbf{L}}\{y,s;p,q\}$
Arbitrary	$X_1=\frac{\partial}{\partial x}$		
	$X_2=\frac{\partial}{\partial t}$	$Z_1=\frac{\partial}{\partial s}$	$\hat{Z}_1=Z_1$
	$X_3=t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}$	$Z_2=y\frac{\partial}{\partial y}+s\frac{\partial}{\partial s}$	$\hat{Z}_2=Z_2$
	$X_4=t\frac{\partial}{\partial x}+\frac{\partial}{\partial v}$	$Z_3=\frac{\partial}{\partial v}$	
	$X_5=x\frac{\partial}{\partial x}+v\frac{\partial}{\partial v}+p\frac{\partial}{\partial p}-\rho\frac{\partial}{\partial \rho}$	$Z_4=v\frac{\partial}{\partial v}+p\frac{\partial}{\partial p}+q\frac{\partial}{\partial q}$	$\hat{Z}_3=p\frac{\partial}{\partial p}+q\frac{\partial}{\partial q}$
	$X_6=p\frac{\partial}{\partial p}+\rho\frac{\partial}{\partial \rho}$	$Z_5=y\frac{\partial}{\partial y}+p\frac{\partial}{\partial p}-q\frac{\partial}{\partial q}$	$\hat{Z}_4=Z_5$
		$Z_6=\frac{\partial}{\partial y}$	$\hat{Z}_5=Z_6$
			$\hat{Z}_6=y^2\frac{\partial}{\partial y}+yp\frac{\partial}{\partial p}-3yq\frac{\partial}{\partial q}$
3	X_1,X_2,X_3,X_4,X_5,X_6	Z_1,Z_2,Z_3,Z_4,Z_5,Z_6	$\hat{Z}_1,\hat{Z}_2,\hat{Z}_3,\hat{Z}_4,\hat{Z}_5,\hat{Z}_6$
	$X_7=xt\frac{\partial}{\partial x}+t^2\frac{\partial}{\partial t}+(x-vt)\frac{\partial}{\partial v}$		$\hat{Z}_7=s^2\frac{\partial}{\partial s}-3sp\frac{\partial}{\partial p}+sq\frac{\partial}{\partial q}$
	$-3tp\frac{\partial}{\partial p}-t\rho\frac{\partial}{\partial \rho}$		
-1	X_1,X_2,X_3,X_4,X_5,X_6	Z_1,Z_2,Z_3,Z_4,Z_5,Z_6	$\hat{Z}_1,\hat{Z}_2,\hat{Z}_3,\hat{Z}_4,\hat{Z}_5,\hat{Z}_6$
		$Z_7=\frac{\partial}{\partial p}+\frac{q}{p}\frac{\partial}{\partial q}$	$\hat{Z}_8=Z_7$
		$Z_8=-s\frac{\partial}{\partial v}+y\frac{\partial}{\partial p}+\frac{yq}{p}\frac{\partial}{\partial q}$	$\hat{Z}_9=y\frac{\partial}{\partial p}+\frac{yq}{p}\frac{\partial}{\partial q}$
			$\hat{Z}_{10}=s\frac{\partial}{\partial p}+\frac{sq}{p}\frac{\partial}{\partial q}$
			$\hat{Z}_{11}=sy\frac{\partial}{\partial p}+\frac{syq}{p}\frac{\partial}{\partial q}$

C. Example: Nonlocal symmetries of polytropic PGD equations

As a given system we consider the Lagrange system $\mathbf{L}\{y,s;v,p,q\}$ (2.6) in the polytropic case $B(p,q)=\gamma p/q$, $\gamma=$const.

In singlet potential system $\mathbf{LW}^4\{y,s;v,p,q,w^4\}$ (2.11) and k-plets including potential equations for w^4, we find that in the polytropic case, up to functional dependence, $S(p,q)=pq^{\gamma}$. In singlet potential system $\mathbf{LW}^5\{y,s;v,p,q,w^5\}$ (2.12) and k-plets including potential equations for w^5, in the polytropic case,

$$K(p,q)=\begin{cases}\dfrac{pq}{\gamma-1}, & \gamma\neq 1\\ pq\ln p, & \gamma=1.\end{cases}$$

We now classify and compare admitted point symmetries of the Lagrange system $\mathbf{L}\{y,s;v,p,q\}$ (2.6), Euler system $\mathbf{E}\{x,t;v,p,\rho\}$ (2.15), singlet potential systems $\mathbf{LW}^1\{y,s;v,p,q,w^1\}$ (2.8), $\mathbf{LW}^2\{y,s;v,p,q,w^2\}$ (2.9), $\mathbf{LW}^3\{y,s;v,p,q,w^3\}$ (2.10), $\mathbf{LW}^4\{y,s;v,p,q,w^4\}$ (2.11), and $\mathbf{LW}^5\{y,s;v,p,q,w^5\}$ (2.12), and subsystems $\underline{\mathbf{L}}\{y,s;p,q\}$ (2.13) and $\underline{\mathbf{LW}}^4\{y,s;p,q,w^4\}$ (2.14). The complete point symmetry classifications of these systems with respect to the polytropic parameter γ are presented in Tables II–IV.

Observe that symmetry $\hat{Z}_7$ is local for systems $\mathbf{E}\{x,t;v,p,\rho\}$, $\underline{\mathbf{L}}\{y,s;p,q\}$, $\mathbf{LW}^1\{y,s;v,p,q,w^1\}$, and $\underline{\mathbf{LW}}^4\{y,s;p,q,w^4\}$ and nonlocal for the other five considered systems; symmetries Z_7 and Z_8 are nonlocal for systems $\mathbf{E}\{x,t;v,p,\rho\}$, $\mathbf{LW}^1\{y,s;v,p,q,w^1\}$, $\mathbf{LW}^3\{y,s;v,p,q,w^3\}$, and $\mathbf{LW}^5\{y,s;v,p,q,w^5\}$ but local for the other five considered systems; symmetries $\hat{Z}_{10}$ and $\hat{Z}_{11}$ are local for the Lagrange subsystem $\underline{\mathbf{L}}\{y,s;p,q\}$ and the subsystem $\underline{\mathbf{LW}}^4\{y,s;p,q,w^4\}$ but nonlocal for the other seven considered systems. Interestingly, the symmetry $\hat{Z}_6$ is local for the Lagrange subsystem $\underline{\mathbf{L}}\{y,s;p,q\}$ for *any* value of the polytropic constant γ, local for the subsystem $\underline{\mathbf{LW}}^4\{y,s;p,q,w^4\}$ *only in the case* $\gamma=1$ (and nonlocal otherwise), but nonlocal for all the other seven considered PGD systems for all values of γ.

TABLE III. Point symmetries of PGD systems $\mathbf{LW}^1\{y,s;v,p,q,w^1\}$, $\mathbf{LW}^2\{y,s;v,p,q,w^2\}$, and $\mathbf{LW}^3\{y,s;v,p,q,w^3\}$ in the polytropic case.

	Admitted point symmetries		
γ	$\mathbf{LW}^1\{y,s;v,p,q,w^1\}$	$\mathbf{LW}^2\{y,s;v,p,q,w^2\}$	$\mathbf{LW}^3\{y,s;v,p,q,w^3\}$
Arbitrary	$I_1=\frac{\partial}{\partial w^1}$ $I_2=Z_1$ $I_3=Z_2+w^1\frac{\partial}{\partial w^1}$ $I_4=Z_3+s\frac{\partial}{\partial w^1}$ $I_5=Z_4+w^1\frac{\partial}{\partial w^1}$ $I_6=Z_5$ $I_7=Z_6$	$J_1=\frac{\partial}{\partial w^2}$ $J_2=Z_1$ $J_3=Z_2+w^2\frac{\partial}{\partial w^2}$ $J_4=Z_3+y\frac{\partial}{\partial w^2}$ $J_5=Z_4+w^2\frac{\partial}{\partial w^2}$ $J_6=Z_5+w^2\frac{\partial}{\partial w^2}$ $J_7=Z_6$ $J_8=\hat{Z}_6+(w^2-yv)\frac{\partial}{\partial v}+yw^2\frac{\partial}{\partial w^2}$	$K_1=\frac{\partial}{\partial w^3}$ $K_2=Z_2+2w^3\frac{\partial}{\partial w^3}$ $K_3=Z_3+ys\frac{\partial}{\partial w^3}$ $K_4=Z_4+w^3\frac{\partial}{\partial w^3}$ $K_5=Z_5+w^3\frac{\partial}{\partial w^3}$
3	$I_1,I_2,I_3,I_4,I_5,I_6,I_7$ $I_8=s^2\frac{\partial}{\partial s}+(w^1-sv)\frac{\partial}{\partial v}$ $-3sp\frac{\partial}{\partial p}+sq\frac{\partial}{\partial q}+sw^1\frac{\partial}{\partial w^1}$	$J_1,J_2,J_3,J_4,J_5,J_6,J_7,J_8$	K_1,K_2,K_3,K_4,K_5
-1	$I_1,I_2,I_3,I_4,I_5,I_6,I_7$	$J_1,J_2,J_3,J_4,J_5,J_6,J_7,J_8$ $J_9=Z_7-s\frac{\partial}{\partial w^2}$ $J_{10}=Z_8-sy\frac{\partial}{\partial w^2}$	K_1,K_2,K_3,K_4,K_5

TABLE IV. Point symmetries of PGD systems $\underline{\mathbf{LW}}^4\{y,s;p,q,w^4\}$, $\mathbf{LW}^4\{y,s;v,p,q,w^4\}$, and $\mathbf{LW}^5\{y,s;v,p,q,w^5\}$ in the polytropic case.

	Admitted point symmetries		
γ	$\underline{\mathbf{LW}}^4\{y,s;p,q,w^4\}$	$\mathbf{LW}^4\{y,s;v,p,q,w^4\}$	$\mathbf{LW}^5\{y,s;v,p,q,w^5\}$
Arbitrary	$\hat{L}_1=\frac{\partial}{\partial w^4}$ $\hat{L}_2=Z_1$ $\hat{L}_3=Z_2+w^4\frac{\partial}{\partial w^4}$ $\hat{L}_4=p\frac{\partial}{\partial p}+q\frac{\partial}{\partial q}+(\gamma+1)w^4\frac{\partial}{\partial w^4}$ $\hat{L}_5=Z_5+(2-\gamma)w^4\frac{\partial}{\partial w^4}$ $\hat{L}_6=Z_6$	$L_1=\hat{L}_1$ $L_2=Z_1$ $L_3=\hat{L}_3$ $L_4=Z_3$ $L_5=v\frac{\partial}{\partial v}+\hat{L}_4$ $L_6=\hat{L}_5$ $L_7=Z_6$	$M_1=\frac{\partial}{\partial w^5}$ $M_2=Z_1$ $M_3=Z_2+w^5\frac{\partial}{\partial w^5}$ $M_4=Z_4+2w^5\frac{\partial}{\partial w^5}$ $M_5=Z_5+w^5\frac{\partial}{\partial w^5}$ $M_6=Z_6$
3	$\hat{L}_1,\hat{L}_2,\hat{L}_3,\hat{L}_4,\hat{L}_5,\hat{L}_6$ $\hat{L}_7=s^2\frac{\partial}{\partial s}-3sp\frac{\partial}{\partial p}+sq\frac{\partial}{\partial q}$	$L_1,L_2,L_3,L_4,L_5,L_6,L_7$	M_1,M_2,M_3,M_4,M_5,M_6
-1	$\hat{L}_1,\hat{L}_2,\hat{L}_3,\hat{L}_4,\hat{L}_5,\hat{L}_6$ $\hat{L}_7=Z_7$ $\hat{L}_8=Z_8$ $\hat{L}_9=\hat{Z}_{10}$ $\hat{L}_{10}=\hat{Z}_{11}$	$L_1,L_2,L_3,L_4,L_5,L_6,L_7$ $L_8=Z_7$ $L_9=Z_8$	M_1,M_2,M_3,M_4,M_5,M_6
1	$\hat{L}_1,\hat{L}_2,\hat{L}_3,\hat{L}_4,\hat{L}_5,\hat{L}_6$ $\hat{L}_{11}=\hat{Z}_6$	$L_1,L_2,L_3,L_4,L_5,L_6,L_7$	M_1,M_2,M_3,M_6 $M_7=Z_4-Z_5+w^5\frac{\partial}{\partial w^5}$

IV. EXACT SOLUTIONS ARISING FROM POTENTIAL SYMMETRIES

Let $\mathbf{S}\{x,t;\mathbf{u},v\}$ (for simplicity, with a single potential variable v) be a potential system of a given system $\mathbf{R}\{x,t;\mathbf{u}\}$. Suppose also that system $\mathbf{S}\{x,t;\mathbf{u},v\}$ admits a point symmetry

$$Y = \xi_S(x,t,\mathbf{u},v)\frac{\partial}{\partial x} + \tau_S(x,t,\mathbf{u},v)\frac{\partial}{\partial t} + \eta_S^j(x,t,\mathbf{u},v)\frac{\partial}{\partial u^j} + \zeta_S^i(x,t,\mathbf{u},v)\frac{\partial}{\partial v^i}, \tag{4.1}$$

which is a nonlocal (potential) symmetry of the given system $\mathbf{R}\{x,t;\mathbf{u}\}$.

A. Computation of exact solutions following from a potential symmetry: The usual approach

A common application of a nonlocal symmetry would be to seek solutions of the potential system $\mathbf{S}\{x,t;\mathbf{u},v\}$ invariant with respect to Y (4.1), and then obtain an exact solution of the given system $\mathbf{R}\{x,t;\mathbf{u}\}$ by projection. (Such exact solutions of $\mathbf{R}\{x,t;\mathbf{u}\}$ often do not arise as solutions invariant with respect to local symmetries of $\mathbf{R}\{x,t;\mathbf{u}\}$.)

The standard algorithm. (Reference 1)

(1) Solve the characteristic system of $m+2$ equations,

$$\frac{dx}{\xi(x,t,\mathbf{u},v)} = \frac{dt}{\tau(x,t,\mathbf{u},v)} = \frac{du^1}{\eta^1(x,t,\mathbf{u},v)} = \cdots = \frac{du^m}{\eta^m(x,t,\mathbf{u},v)} = \frac{dv}{\zeta(x,t,\mathbf{u},v)} \tag{4.2}$$

The $m+2$ invariants of the symmetry (4.1) are constants of integration of the characteristic system (4.2); denote them

$$z = Z(x,t,\mathbf{u},v), \quad h_1 = H_1(x,t,\mathbf{u},v), \ldots, \quad h_{m+1} = H_{m+1}(x,t,\mathbf{u},v). \tag{4.3}$$

(2) Find the translated coordinate $\hat{z}=\hat{Z}(x,t,\mathbf{u},v)$, which is a solution of $X\hat{z}=1$. Variables $\{z,\hat{z},h_1(z,\hat{z}),\ldots,h_{m+1}(z,\hat{z})\}$ are *canonical coordinates*, in which the symmetry Y (4.1) becomes the translation symmetry $Y=\partial/\partial\hat{z}$.

(3) Express the problem variables in terms of the canonical coordinates,

$$\begin{aligned} x &= x(z,\hat{z},h_1(z,\hat{z}),\ldots,h_{m+1}(z,\hat{z})), \\ t &= t(z,\hat{z},h_1(z,\hat{z}),\ldots,h_{m+1}(z,\hat{z})), \\ u^i &= u^i(z,\hat{z},h_1(z,\hat{z}),\ldots,h_{m+1}(z,\hat{z})), \quad i=1,\ldots,m, \\ v &= v(z,\hat{z},h_1(z,\hat{z}),\ldots,h_{m+1}(z,\hat{z})). \end{aligned} \tag{4.4}$$

In the potential system $\mathbf{S}\{x,t;\mathbf{u},v\}$, perform a local change of variables,

$$(x,t;u^1(x,t),\ldots,u^m(x,t),v(x,t)) \to (z,\hat{z};h_1(z,\hat{z}),\ldots,h_{m+1}(z,\hat{z})), \tag{4.5}$$

to obtain a locally equivalent system $\widetilde{\mathbf{S}}\{z,\hat{z};h_1,\ldots,h_{m+1}\}$.

(4) Assume the independence of $h_1,\ldots,h_{m+1}$ on the translation variable $\hat{z}$. Solve the resulting ordinary differential equations to obtain $h_1(z),\ldots,h_{m+1}(z)$.

(5) Using (4.4), express the solution $(\mathbf{u}(x,t),v(x,t))$ of the potential system $\mathbf{S}\{x,t;\mathbf{u},v\}$. The vector $\mathbf{u}(x,t)$ is the desired solution of the given PDE system $\mathbf{R}\{x,t;\mathbf{u}\}$.

Remark 2. The above procedure directly generalizes to the case of $N\geq 3$ independent variables.

B. Computation of exact solutions following from a potential symmetry: The extended procedure

1. First extension

Pucci and Saccomandi[20] argued that for some examples one might obtain a broader class of exact solutions if, instead of the potential system $\mathbf{S}\{x,t;\mathbf{u},v\}$, the substitution of canonical coordinates and the consequent symmetry reduction are instead performed for the *given system* $\mathbf{R}\{x,t;\mathbf{u}\}$. This modification of steps 3 and 4 in Sec. IV A is equivalent to requiring that the potential variable v is sought in the invariant form, but does not have to satisfy the potential equations.

It is not clear how useful this extension could be. To answer this question, one would need to consider more examples. In particular, it is necessary to study whether or not additional solutions arising from using this method are obtainable from other local symmetries using only the usual ansatz (Sec. IV A) For examples considered in this paper (Sec. V below), the above "first extension" alone did not yield any new solutions.

2. Second extension

Another extension idea is due to Sjöberg and Mahomed; in Ref. 9 they suggested to seek solutions $(\mathbf{u}(x,t),v(x,t))$ of the potential system $\mathbf{S}\{x,t;\mathbf{u},v\}$ where $v(x,t)$ is not necessarily an *invariant* solution.

It is also unclear whether or not this extension alone could yield new exact solutions of the given system. In particular, one can show that for the solution derived in Ref. 9 this extension was not necessary (one could have used the usual method). For PGD equations considered in the present paper (Sec. V below), the "second extension" alone did not yield any new exact solutions either. However, the idea, in general, appears to be useful.

3. The combined approach

In the present paper, using the example of nonlinear polytropic PGD equations, we show that the above extension ideas are indeed useful when used *together*. As a result, one obtains genuinely new exact solutions of a given PDE system: such solutions do not arise as invariant solutions of the given system with respect to its point symmetries, nor do they arise as invariant solutions of the corresponding potential system!

Incorporating the extension ideas from Secs. IV B 1 and IV B 2, we formulate an extended algorithm for finding exact solutions of a PDE system following from an admitted nonlocal (potential) symmetry.

The extended algorithm.

(1) Solve the characteristic system of $m+2$ equations

$$\frac{dx}{\xi(x,t,\mathbf{u},v)}=\frac{dt}{\tau(x,t,\mathbf{u},v)}=\frac{du^1}{\eta^1(x,t,\mathbf{u},v)}=\cdots=\frac{du^m}{\eta^m(x,t,\mathbf{u},v)}=\frac{dv}{\zeta(x,t,\mathbf{u},v)}. \quad (4.6)$$

The $m+2$ invariants of the symmetry Y (4.1) are constants of integration of the characteristic system (4.2); denote them

$$z=Z(x,t,\mathbf{u},v), \quad h_1=H_1(x,t,\mathbf{u},v), \ldots, \quad h_{m+1}=H_{m+1}(x,t,\mathbf{u},v). \quad (4.7)$$

(2) Find the "translation coordinate" $\hat{z}=\hat{Z}(x,t,\mathbf{u},v)$, which is a solution of $X\hat{z}=1$. Variables $\{z,\hat{z},h_1(z,\hat{z}),\ldots,h_{m+1}(z,\hat{z})\}$ are canonical coordinates, in which the symmetry Y (4.1) becomes the translation symmetry $Y=\partial/\partial\hat{z}$.

(3) Express the problem variables in terms of the canonical coordinates,

$$x=x(z,\hat{z},h_1(z,\hat{z}),\ldots,h_{m+1}(z,\hat{z})),$$

$$t = t(z,\hat{z},h_1(z,\hat{z}), \ldots ,h_{m+1}(z,\hat{z})),$$

$$u^i = u^i(z,\hat{z},h_1(z,\hat{z}), \ldots ,h_{m+1}(z,\hat{z})), \quad i = 1, \ldots ,m, \tag{4.8}$$

$$v = v(z,\hat{z},h_1(z,\hat{z}), \ldots ,h_{m+1}(z,\hat{z})).$$

In the *given system* $\mathbf{R}\{x,t;\mathbf{u}\}$, perform the local change of variables,

$$(x,t;u^1(x,t), \ldots ,u^m(x,t)) \to (z,\hat{z};h_1(z,\hat{z}), \ldots ,h_{m+1}(z,\hat{z})). \tag{4.9}$$

(4) Assume that *one or more of the functions* h_i that participate in the expression for the potential variable v in (4.4), *depend on both* z and $\hat{z}$, whereas all other functions h_i $(1 \le i \le m)$ only depend on z. Solve the resulting *PDEs* to find all functions h_i. Using (4.4), express $\mathbf{u}(x,t), v(x,t)$. The vector $\mathbf{u}(x,t)$ is a solution of the PDE system $\mathbf{R}\{x,t;\mathbf{u}\}$; the pair $(\mathbf{u}(x,t), v(x,t))$ is generally *not a solution* of the potential system $\mathbf{S}\{x,t;\mathbf{u},v\}$.

(5) Using (4.4), express $(\mathbf{u}(x,t), v(x,t))$. Here $\mathbf{u}(x,t)$ is the solution of the given PDE system $\mathbf{R}\{x,t;\mathbf{u}\}$.

In other words, the extended algorithm is different from the usual one in two ways: the potential(s) $v(x,t)$ do not have to be a part of the solution of the potential system $\mathbf{S}\{x,t;\mathbf{u},v\}$ and do not have to be invariant under the symmetry Y (4.1). [However, $\mathbf{u}(x,t)$ is indeed a solution to the given system $\mathbf{R}\{x,t;\mathbf{u}\}$!]

V. EXAMPLES OF USING THE EXTENDED PROCEDURE FOR EXACT SOLUTION COMPUTATION

A. Example 1

Here we consider the Lagrange PGD system $\mathbf{L}\{y,s;v,p,q\}$ (2.6) in a particular polytropic case $B(p,q) = 3p/q$ $(\gamma = 3)$. In this case, it admits a nonlocal symmetry

$$\mathrm{I}_8 = s^2\frac{\partial}{\partial s} + (w^1 - sv)\frac{\partial}{\partial v} - 3sp\frac{\partial}{\partial p} + sq\frac{\partial}{\partial q} + sw^1\frac{\partial}{\partial w^1}, \tag{5.1}$$

which is a point symmetry of the potential system $\mathbf{LW}^1\{y,s;v,p,q,w^1\}$ (2.8) (see Table III).

The five invariants of the symmetry (5.1) are given by

$$z = y, \quad h_1 = s^3 p, \quad h_2 = \frac{q}{s}, \quad h_3 = \frac{w^1}{s}, \quad h_4 = sv - w^1. \tag{5.2}$$

As a translated canonical coordinate $\hat{z}$, here we choose

$$\hat{z} = 1/s. \tag{5.3}$$

1. Solutions obtained using the usual algorithm

Following the standard algorithm of Sec. IV A above, we first seek the solution of the potential system $\mathbf{LW}^1\{y,s;v,p,q,w^1\}$ (2.8) invariant with respect to I_8 (5.1). We assume $h_i = h_i(z)$ and express

$$p(y,s) = \frac{h_1(y)}{s^3}, \quad q(y,s) = sh_2(y), \quad v(y,s) = \frac{h_4(y)}{s} + h_3(y), \quad w^1(y,s) = sh_3(y) \tag{5.4}$$

Substitution of (5.4) into $\mathbf{LW}^1\{y,s;v,p,q,w^1\}$ (2.8) yields

$$h_1'(y) = 0, \quad h_3'(y) = h_2(y), \quad h_4(y) = 0, \quad h_1(y)h_4'(y) = 0, \quad h_1(y)h_2(y) = h_1(y)h_3'(y),$$

with the solution $(h_1 \neq 0)$ given by

$$h_1(y)=C,\quad h_2(y)=h_3'(y),\quad h_3(y)=f(y),\quad h_4(y)=0,$$

where $f(y)$ is an arbitrary function and C is an arbitrary constant.

The corresponding family of solutions $(v(y,s),p(y,s),q(y,s),w^1(y,s))$ of the potential system $\mathbf{LW}^1\{y,s;v,p,q,w^1\}$ (2.8) is found from (5.4). In particular, solutions $(v(y,s),p(y,s),q(y,s))$ of the Lagrange system $\mathbf{L}\{y,s;v,p,q\}$ (2.6) are given by

$$v(y,s)=f(y),\quad p(y,s)=\frac{C}{s^3},\quad q(y,s)=sf'(y). \tag{5.5}$$

2. Solutions obtained using the extended algorithm

Using the extended algorithm of Sec. IV B 3, in (5.4), we assume that the expression for the potential variable may, in fact, be noninvariant: $h_3(y,\hat{z})=h_3(y,s)$. Therefore

$$p(y,s)=\frac{h_1(y)}{s^3},\quad q(y,s)=sh_2(y),\quad v(y,s)=\frac{h_4(y)}{s}+h_3(y,s),\quad w^1(y,s)=sh_3(y,s). \tag{5.6}$$

Substitution of (5.6) into the given system $\mathbf{L}\{y,s;v,p,q\}$ (2.6) yields two PDEs

$$sh_2(y)-h_4'(y)=s\frac{\partial}{\partial y}h_3(y,s),\quad s^3\frac{\partial}{\partial s}h_3(y,s)+h_1'(y)=-sh_4(y). \tag{5.7}$$

The solution of (5.7) is

$$h_1(y)=P1(y)=2C_1y+C_2,\quad h_2(y)=f_1'(y),\quad h_3(y)=f_1(y)+\frac{C_1}{s^2}+\frac{f_2(y)}{s},\quad h_4(y)=-f_2(y),$$

where $f_1(y),f_2(y)$ are sufficiently smooth arbitrary functions and C_1,C_2 are arbitrary constants.

The corresponding family of solutions $(v(y,s),p(y,s),q(y,s))$ of the Lagrange PGD system $\mathbf{L}\{y,s;v,p,q\}$ (2.6) is given by

$$v(y,s)=f_1(y)+\frac{C_1}{s^2},\quad p(y,s)=\frac{2yC_1+C_2}{s^3},\quad q(y,s)=sf_1'(y) \tag{5.8}$$

and generalizes solutions (5.5). The corresponding potential $w^1(y,s)$ is given by

$$w^1(y,s)=sf_1(y)+\frac{C_1}{s}+f_2(y). \tag{5.9}$$

The following two theorems address the obtainability of the family of solutions (5.8) using simpler methods.

Theorem 1. *The exact solutions* (5.8) *of the Lagrange PGD system* $\mathbf{L}\{y,s;v,p,q\}$ (2.6) *do not arise as invariant solutions of* $\mathbf{L}\{y,s;v,p,q\}$ or as invariant solutions of the potential system $\mathbf{LW}^1\{y,s;v,p,q,w^1\}$ (2.8), *with respect to any of their point symmetries.*

The proof of Theorem 1 is given in Appendix.

Theorem 2. *The exact solutions* (5.8) *do not arise from extensions described in* Secs. IV B 1 and IV B 2, *but only from the combined extended algorithm described in* Sec. IV B 3.

The proof of Theorem 2 proceeds by a direct computation.

B. Example 2

For the second example, we consider the Lagrange PGD system $\mathbf{L}\{y,s;v,p,q\}$ (2.6) in the general polytropic case $B(p,q)=\gamma p/q,\ \gamma\in\mathbb{R}$. It admits a nonlocal (potential) symmetry,

$$\mathrm{J}_8 = y^2\frac{\partial}{\partial y} + yp\frac{\partial}{\partial p} - 3yq\frac{\partial}{\partial q} + (w^2 - yv)\frac{\partial}{\partial v} + yw^2\frac{\partial}{\partial w^2}, \tag{5.10}$$

which is a point symmetry of the potential system $\mathbf{LW}^2\{y,s;v,p,q,w^2\}$ (2.9) (see Table III).

The five invariants of the symmetry (5.10) are

$$z = s, \quad h_1 = \frac{p}{y}, \quad h_2 = y^3 q, \quad h_3 = \frac{w^2}{y}, \quad h_4 = yv - w^2. \tag{5.11}$$

As a translated canonical coordinate $\hat{z}$, here we choose

$$\hat{z} = 1/y. \tag{5.12}$$

1. Solutions obtained using the usual algorithm

Following the standard algorithm (Sec. IV A) we start from seeking the solution of the potential system $\mathbf{LW}^2\{y,s;v,p,q,w^2\}$ (2.9) invariant with respect to J_8 (5.10). We assume $h_i = h_i(z)$ and express

$$p(y,s) = yh_1(s), \quad q(y,s) = \frac{h_2(s)}{y^3}, \quad v(y,s) = \frac{h_4(s)}{y} + h_3(s), \quad w^2(y,s) = yh_3(s). \tag{5.13}$$

Substitution of (5.13) into the potential system $\mathbf{LW}^2\{y,s;v,p,q,w^2\}$ (2.9) yields the equations

$$h_2'(y) = 0, \quad h_3'(s) = -h_1(s), \quad h_4(y) = 0, \quad h_1'(y)h_2(y) = 0, \quad \gamma h_1(y)h_4 = 0,$$

with the solution

$$h_1(y) = C_1, \quad h_2(y) = C_2, \quad h_3(y) = -C_1 s + C_3, \quad h_4(y) = 0, \tag{5.14}$$

where C_1, C_2, C_3 are arbitrary constants.

The corresponding family of invariant solutions $(v(y,s), p(y,s), q(y,s))$ of the Lagrange PGD system $\mathbf{L}\{y,s;v,p,q\}$ (2.6) is given by

$$v(y,s) = -C_1 s + C_3, \quad p(y,s) = C_1 y, \quad q(y,s) = \frac{C_2}{y^3}. \tag{5.15}$$

2. Solutions obtained using the extended algorithm

Using the extended algorithm of Sec. IV B 3, we let

$$p(y,s) = yh_1(s), \quad q(y,s) = \frac{h_2(s)}{y^3}, \quad v(y,s) = \frac{h_4(s)}{y} + h_3(y,s), \quad w^2(y,s) = yh_3(y,s). \tag{5.16}$$

Substitution of (5.6) into the *given* system $\mathbf{L}\{y,s;v,p,q\}$ (2.6) yields two PDEs,

$$h_2'(s) + yh_4(s) = y^3\frac{\partial}{\partial y}h_3(y,s), \quad h_4'(s) + y\frac{\partial}{\partial s}h_3(y,s) + yh_1(s) = 0,$$
$$\gamma y h_1(s)h_4(s) = h_1'(s)h_2(s) + \gamma y^3 h_1(s)\frac{\partial}{\partial y}h_3(y,s). \tag{5.17}$$

The PDE system (5.17) admits solutions that lead to *three families of solutions* of the Lagrange PGD system $\mathbf{L}\{y,s;v,p,q\}$ (2.6).

The first family is given by

$$\mathcal{F}_1: \quad v(y,s) = -a_1 s + a_3, \quad p(y,s) = a_1 y, \quad q(y,s) = \frac{a_2}{y^3} \tag{5.18}$$

and coincides with solutions (5.15) obtained using a usual method.

The second family is given by

$$\mathcal{F}_2: \quad v(y,s) = \frac{b_1}{y^2} + b_2, \quad p(y,s) = 0, \quad q(y,s) = \frac{-2b_1 s + b_3}{y^3}. \tag{5.19}$$

The third family (holding only for integer $\gamma = n$) is given by

$$\mathcal{F}_3: \quad \begin{cases} v(y,s) = \dfrac{c_1 n^n (-1)^{n-1}}{n-1}(s+c_2)^{1-n} + c_3 - \dfrac{c_4}{y^2} \\ p(y,s) = c_1 n^n (-1)^{n-1}(s+c_2)^{-n} y \\ q(y,s) = \dfrac{2c_4(s+c_2)}{y^3} \end{cases} \tag{5.20}$$

for $n \neq 1$, and by

$$\mathcal{F}_3': \quad \begin{cases} v(y,s) = -\dfrac{1}{c_1}\ln(s+c_2) + c_3 - \dfrac{c_4}{y^2} \\ p(y,s) = \dfrac{y}{c_1(s+c_2)} \\ q(y,s) = \dfrac{2c_4(s+c_2)}{y^3} \end{cases} \tag{5.21}$$

for $n=1$. [In (5.18)–(5.21), a_i, b_j, c_k are arbitrary constants.]

The corresponding expressions for the potential w^2 for the solution families (5.18)–(5.21) are

$$\mathcal{F}_2: w^2 = b_2 y + \frac{b_1}{y} + f(s),$$

$$\mathcal{F}_3: w^2 = \frac{c_1 n^n (-1)^{n-1}}{n-1}(s+c_2)^{1-n} y + c_3 y - \frac{c_4}{y} + f(s), \tag{5.22}$$

$$\mathcal{F}_3': w^2 = -\frac{y}{c_1}\ln(s+c_2) + c_3 y - \frac{c_4}{y} + f(s),$$

where $f(s)$ is an arbitrary function.

The following two theorems demonstrate that the new families of solutions obtainability of the family of solutions $\mathcal{F}_2$ (5.19) and $\mathcal{F}_3, \mathcal{F}_3'$ (5.20), (5.21) cannot be obtained using a simpler ansatz.

Theorem 3. *Families of exact solutions (5.19), (5.20), and (5.21) of the polytropic Lagrange PGD system* $\mathbf{L}\{y,s;v,p,q\}$ (2.6) *do not arise as invariant solutions of* $\mathbf{L}\{y,s;v,p,q\}$ *or as invariant solutions of the potential system* $\mathbf{LW}^2\{y,s;v,p,q,w^1\}$ (2.9), *with respect to any of their point symmetries.*

The proof of Theorem 3 is presented in Appendix.

Theorem 4. *Families of exact solutions (5.19), (5.20), and (5.21) do not arise from extensions described in* Secs. IV B 1 and IV B 2, *but only from the combined extended algorithm described in* Sec. IV B 3.

The proof of Theorem 4 proceeds by a direct computation.

VI. CONCLUSIONS AND DISCUSSION

A nonlocal (potential) symmetry admitted by a given PDE system may be used to seek exact solutions of that system. The usual way to do it consists in finding invariant solution of the corresponding potential system and projecting it on the space of variables of the given one. In this paper, we introduced an extended algorithmic procedure that generalizes the usual algorithm and is capable of generating additional exact solutions of the given system. In particular, the extended algorithm is different from the usual one in two ways: (i) potential variable(s) do not have to solve the potential system and (ii) they do not have to be invariant under the action of the symmetry.

As an example, we considered the adiabatic Lagrange system $\mathbf{L}\{y,s;v,p,q\}$ (2.6) of PGD equations. In Sec. II E, we derived five local conservation laws of $\mathbf{L}\{y,s;v,p,q\}$ and constructed a tree of equivalent nonlocally related PDE systems. In Sec. III C, in the case of a polytropic gas with arbitrary polytropic exponent γ, we completely classified admitted point symmetries of the Lagrange system $\mathbf{L}\{y,s;v,p,q\}$ (2.6), its singlet potential systems, and their nonlocally related subsystems, extending work in Refs. 4, 9, and 13 and references therein. As a result, we obtained a number of nonlocal symmetries admitted by the system $\mathbf{L}\{y,s;v,p,q\}$ (2.6), holding for general and specific values of the parameter γ.

In Sec. V, we considered nonlocal symmetries I_8 (5.1) and J_8 (5.10) of $\mathbf{L}\{y,s;v,p,q\}$ (2.6) and used them to construct exact solutions using a usual invariant solution algorithm and the new extended algorithm. It was shown that the extended ansatz yields additional families of exact solutions and proved that these families do not arise as invariant solutions of the given system or a corresponding potential system with respect to their point symmetries. Moreover, these additional families of solutions cannot be obtained using simpler extension ideas suggested in previous literature.[9,20]

It can be shown, in general, that the set of solutions found from the extended procedure always includes all solutions found using the usual algorithm. Moreover, the amount of computations involved in using the extended procedure does not significantly exceed that for the usual one. We therefore conclude that the extended algorithm presented in this paper should be adopted as a standard procedure for finding exact solutions from admitted potential symmetries.

As seen in many examples in literature and in examples in this paper, nonlocal symmetries of a given often arise as symmetries of nonlocally related subsystems in the tree [see symmetry classifications of subsystems $\underline{\mathbf{L}}\{y,s;p,q\}$ (Table II) and $\underline{\mathbf{LW}}^4\{y,s;p,q,w^4\}$ (Table IV)]. In the future work, it is important to study whether it is possible to develop an optimal procedure of using such nonlocal symmetries for the construction of exact solutions of a PDE system of interest.

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APPENDIX

1. Proofs of theorems 1 and 3

Proof of Theorem 1.

(a) By a direct computation, it is easy to check that (5.8) is a solution of $\mathbf{L}\{y,s;v,p,q\}$ (2.6), but (5.8), (5.9) *is not a solution* of the potential system $\mathbf{LW}^1\{y,s;v,p,q,w^1\}$ (2.8) for any $f_2(y)$, therefore (5.8) and (5.9) cannot arise as an invariant solution of the potential system $\mathbf{LW}^1\{y,s;v,p,q,w^1\}$ with respect to its point symmetries.

(b) We now show that (5.8) is not an invariant solution of the Lagrange system $\mathbf{L}\{y,s;v,p,q\}$ (2.6). Consider a general linear combination of point symmetry generators

$$\mathrm{Z} = \sum_{i=1}^{6} a_i \mathrm{Z}_i \tag{A1}$$

admitted by $\mathbf{L}\{y,s;v,p,q\}$ (2.6) (see Table II). A solution (5.8) is an invariant solution if and only if there exists such a nontrivial set of coefficients $\{a_i\}_{i=1}^{6}$ that

$$\mathrm{Z}(v - v(y,s)) \equiv 0, \quad \mathrm{Z}(p - p(y,s)) \equiv 0, \quad \mathrm{Z}(q - q(y,s)) \equiv 0$$

simultaneously, where $v(y,s), p(y,s), q(y,s)$ are given by (5.8). One can check that this is the case only when $a_i = 0$, $i = 1, \ldots, 6$, i.e., (5.8) is *not* an invariant solution of the Lagrange system $\mathbf{L}\{y,s;v,p,q\}$. □

Proof of Theorem 5.37.

(a) Again by a direct computation, one shows that families of solutions (5.19)–(5.21) solve the Lagrange system $\mathbf{L}\{y,s;v,p,q\}$ (2.6), but the same families [with potentials (5.22)] *do not solve* the potential system $\mathbf{LW}^2\{y,s;v,p,q,w^2\}$ (2.9). Thus solutions (5.19)–(5.21) cannot arise as invariant solutions of the potential system $\mathbf{LW}^2\{y,s;v,p,q,w^2\}$ with respect to any of its point symmetries.

(b) Consider a general linear combination of point symmetry generators (A1) admitted by the Lagrange system $\mathbf{L}\{y,s;v,p,q\}$ (2.6). For solutions from the families (5.19)–(5.21), one can check that

$$\mathrm{Z}(v - v(y,s)) \equiv 0, \quad \mathrm{Z}(p - p(y,s)) \equiv 0, \quad \mathrm{Z}(q - q(y,s)) \equiv 0$$

if and only if $a_1 = \ldots = a_6 = 0$, which means that solutions (5.19)–(5.21) are not invariant solutions of $\mathbf{L}\{y,s;v,p,q\}$ under its admitted point symmetries. □

[1] G. W. Bluman and S. Kumei, *Symmetries and Differential Equations* (Springer-Verlag, New York, 1989).
[2] P. J. Olver, *Applications of Lie Groups to Differential Equations* (Springer, New York, 1986).
[3] G. Bluman and A. F. Cheviakov, J. Math. Phys. **46**, 123506 (2005).
[4] G. Bluman, A. F. Cheviakov, and N. M. Ivanova, J. Math. Phys. **47**, 113505 (2006).
[5] G. Bluman and A. F. Cheviakov, J. Math. Anal. Appl. **333**, 93 (2007).
[6] G. Bluman and P. Doran-Wu, Acta Appl. Math. **41**, 21 (1995).
[7] N. M. Ivanova and R. O. Popovych, J. Math. Phys. **46**, 043502 (2005).
[8] G. Bluman, A. F. Cheviakov, and J.-F. Ganghoffer, "Nonlocally related PDE systems for one-dimensional nonlinear elastodynamics," J. Eng. Math. (to be published).
[9] A. Sjöberg and F. M. Mahomed, Appl. Math. Comput. **150**, 379 (2004).
[10] S. Anco and G. Bluman, J. Math. Phys. **38**, 3508 (1997).
[11] O. I. Bogoyavlenskij, Phys. Lett. A **291**, 256 (2001).
[12] F. Galas, Physica D **63**, 87 (1993).
[13] S. Akhatov, R. Gazizov, and N. Ibragimov, J. Sov. Math. **55**, 1401 (1991).
[14] G. Bluman, J. Math. Phys. **46**, 023505 (2005).
[15] A. F. Cheviakov, Comput. Phys. Commun. **176**, 48 (2007) The GEM package and documentation is available at http://www.math.ubc.ca/~alexch/gem/
[16] S. Anco and G. Bluman, Phys. Rev. Lett. **78**, 2869 (1997).
[17] S. Anco and G. Bluman, Eur. J. Appl. Math. **13**, 567 (2002).
[18] T. Wolf, Eur. J. Appl. Math. **13**, 129 (2002).
[19] I. M. Anderson and C. G. Torre, e-print arXiv:hep-th/9608008v2.
[20] E. Pucci and G. Saccomandi, J. Phys. A **26**, 681 (1993).