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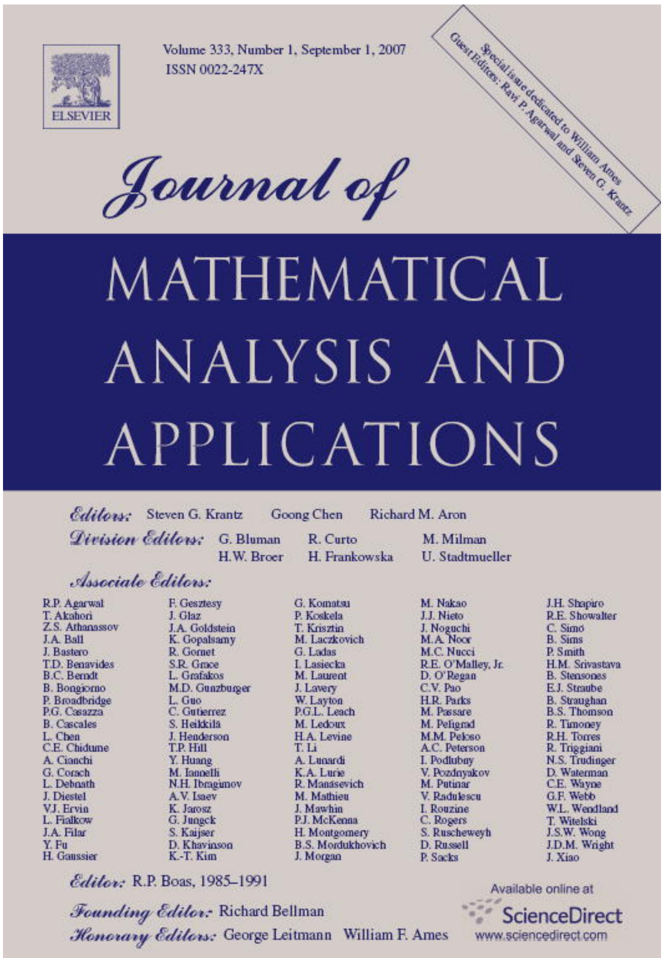

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Nonlocally related systems, linearization and nonlocal symmetries for the nonlinear wave equation

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Dedicated to Bill Ames on his 80th birthday

Abstract

The nonlinear wave equation $u_{tt} = (c^2(u)u_x)_x$ arises in various physical applications. Ames et al. [W.F. Ames, R.J. Lohner, E. Adams, Group properties of $u_{tt} = [f(u)u_x]_x$, Int. J. Nonlin. Mech. 16 (1981) 439–447] did the complete group classification for its admitted point symmetries with respect to the wave speed function $c(u)$ and as a consequence constructed explicit invariant solutions for some specific cases. By considering conservation laws for arbitrary $c(u)$, we find a tree of nonlocally related systems and subsystems which include related linear systems through hodograph transformations. We use existing work on such related linear systems to extend the known symmetry classification in [W.F. Ames, R.J. Lohner, E. Adams, Group properties of $u_{tt} = [f(u)u_x]_x$, Int. J. Nonlin. Mech. 16 (1981) 439–447] to include nonlocal symmetries. Moreover, we find sets of $c(u)$ for which such nonlinear wave equations admit further nonlocal symmetries and hence significantly further extend the group classification of the nonlinear wave equation.
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1. Introduction

For any given system of partial differential equations (PDEs), one can systematically construct nonlocally related potential systems and subsystems [2,3] having the same solution set as

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the given system. Analysis of such trees of nonlocally related systems can yield new analytical results for the given system. In particular, using this approach, one can systematically calculate nonlocal symmetries and conservation laws, construct invariant and nonclassical solutions, as well as obtain linearizations, etc. [2,3]. Moreover, since all such related systems contain all solutions of the given system, any general method of analysis (qualitative, numerical, perturbation, conservation laws, etc.) considered for a given PDE system may be tried again on any nonlocally related potential system or subsystem. In this way, new results may be obtained for any method of analysis that is not coordinate-dependent since the systems within a tree are related in a nonlocal manner.

The nonlinear wave equation

$$\mathbf{U}\{x,t,u\}=0: \quad u_{tt}=\left(c^2(u)u_x\right)_x \tag{1.1}$$

and the linear wave equation with variable wave speed

$$\mathbf{L}\{x,t,u\}=0: \quad u_{tt}=d^2(x)u_{xx} \tag{1.2}$$

are important equations that arise in various physical problems in the context of oscillations and wave phenomena.

The equations **U** (1.1) and **L** (1.2) cannot be directly related by a point transformation. In this paper, we consider these equations within the framework of nonlocally related PDE systems [2,3], and demonstrate that for an arbitrary smooth nonlinearity $c(u)$ there exists a nonlocal transformation that relates the nonlinear wave equation **U** (1.1) to a linear wave equation **L** (1.2), such that to every solution of **U** there is a corresponding solution of **L**, and vice versa. Thus the nonlinear wave equation **U** (1.1) is linearizable by a nonlocal transformation.

The second goal of this paper is to study symmetries of the nonlinear wave equation **U** (1.1). A *symmetry* of a system of differential equations is defined topologically as any transformation of its solution manifold into itself. Hence, symmetry transformations are not restricted to local transformations acting on the given system's dependent and independent variables. Many examples demonstrate that local symmetries of the given system computed directly by Lie's algorithm do not include all calculable symmetries of a given system. Indeed, other (nonlocal) symmetries may be found if one applies Lie's algorithm to DE systems nonlocally related to the given system.

Point symmetries of the nonlinear wave equation **U** (1.1) were classified by Ames et al. [1]. In this paper we extend this work and classify *nonlocal* symmetries of the equation **U** (1.1), for various forms of the nonlinearity $c(u)$. We construct a tree of nonlocally related potential systems and subsystems for the nonlinear wave equation **U**, and correspondingly find nonlocal symmetries that arise as point symmetries of other equations in the tree.

This paper is organized as follows. In Section 2, we construct a tree of nonlocally related potential systems and subsystems for the nonlinear wave equation **U**. In particular, we demonstrate that for any $c(u)$ this equation is linearizable by a nonlocal transformation, and establish a nonlocal relation between equations **U** and **L**.

In Section 3, we consider point symmetries of PDE systems nonlocally related to **U** and compare them to point symmetries of **U** found by Ames et al. [1]. In particular, we make use of the results in [4] on symmetry analysis of the linear wave equation **L** (1.2) and its potential system, and results in [6] on symmetry analysis of some other nonlocally related systems within the tree to obtain nonlocal symmetries of **U**. Symmetry analysis of some nonlocally related systems that were not considered before is shown to yield further nonlocal symmetries of **U**.

In Section 4, we discuss the obtained results and make some concluding remarks.

In this work, a package `GeM` for Maple [8] is used for the automated symmetry analysis and classifications.

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2. Nonlocal analysis of the nonlinear wave equation

2.1. *Linearizability of the nonlinear wave equation* (1.1) *through a nonlocal transformation*

Theorem 1. *There exists a linear PDE system* **XT** *which is nonlocally related to the nonlinear wave equation* **U** (1.1)*, and whose solution set is equivalent to the solution set of the nonlinear wave equation* **U**. *Thus the nonlinear wave equation* **U** (1.1) *is linearizable by a nonlocal transformation.*

Proof. The nonlinear wave equation **U** (1.1) is a divergence expression, and hence one can introduce a potential v, to obtain a potential system

$$\mathbf{UV}\{x,t,u,v\}=0: \quad \begin{cases} u_t - v_x = 0, \\ v_t - c^2(u)u_x = 0, \end{cases} \tag{2.3}$$

which is equivalent but nonlocally related to **U**.

The system **UV** (2.3) has two dependent variables (u, v) and two independent variables (x, t). From its form it is evident that (2.3) is linearizable by a one-to-one hodograph transformation (see e.g. [9]). In particular, after a point transformation

$$x = x(u,v), \quad t = t(u,v), \tag{2.4}$$

the system (2.3) becomes

$$\mathbf{XT}\{x,t,u,v\}=0: \quad \begin{cases} x_v - t_u = 0, \\ x_u - c^2(u)t_v = 0, \end{cases} \tag{2.5}$$

which is a *linear PDE system* with variable coefficients. □

2.2. *Nonlocally related systems for the nonlinear wave equation* **U** (1.1)

For any given PDE system, the algorithm for construction of a *tree of nonlocally related PDE systems* was developed in [2,3]. First, one constructs the local conservation laws of the given system. Each local conservation law yields a set of potential equations, involving new nonlocal variables (potentials). Using the conservation laws individually, as pairs, as triplets, ..., as an n-plet, (n being the number of conservation laws), one constructs all possible $(2^n - 1)$ potential systems of the given system arising from its local conservation laws. *Subsystems* are obtained by excluding one or more dependent variables from potential systems, including exclusions obtained through an interchange of a dependent and an independent variable. After eliminating locally related and redundant systems, one obtains a tree of nonlocally related PDE systems for a given PDE system. Using linearly independent nonlocal conservation laws found for the $2^n - 1$ potential systems and thereby introducing further potentials, one can obtain an *extended tree of nonlocally related tree of PDE systems* [2,3].

We now construct nonlocally related PDE systems for the nonlinear wave equation **U** (1.1) that arise from its known local conservation laws.

(I). Conservation laws of the nonlinear wave equation U (1.1).

We look for conservation laws of the nonlinear wave equation (1.1) in the form of divergence expressions

$$D_t T(x,t,u,u_x,u_t) + D_x X(x,t,u,u_x,u_t) = 0, \tag{2.6}$$

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using multipliers that depend on dependent and independent variables. (For detailed description of algorithms for constructing conservation laws, see [7].) We consider multipliers in the form $\Lambda = \Lambda(x,t,\tilde{u})$, such that the product of the multiplier and Eq. (1.1) is a divergence expression

$$\Lambda(x,t,\tilde{u})\mathbf{U}\{x,t,\tilde{u}\} = D_t T(x,t,\tilde{u},\partial\tilde{u}) + D_x X(x,t,\tilde{u},\partial\tilde{u}), \tag{2.7}$$

for an arbitrary function $\tilde{u}$. Such multipliers, on the solution space $\tilde{u} = u$, yield conservation laws (2.6) of the given nonlinear wave equation.

The solution of the conservation law determining equations that follow from applying the Euler operator with respect to $\tilde{u}$ to (2.7) yields the following result.

Theorem 2. *For an arbitrary wave speed $c(u)$, the nonlinear wave equation* **U** (1.1) *possesses local conservation laws*

$$D_t\big(x[tu_t - u]\big) - D_x\left(t\left[xc^2(u)u_x - \int c^2(u)\,du\right]\right) = 0, \tag{2.8}$$

$$D_t(xu_t) - D_x\left(xc^2(u)u_x - \int c^2(u)\,du\right) = 0, \tag{2.9}$$

$$D_t(tu_t - u) - D_x\big(tc^2(u)u_x\big) = 0, \tag{2.10}$$

$$D_t(u_t) - D_x\big(c^2(u)u_x\big) = 0, \tag{2.11}$$

which correspond to multipliers $\{\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4\} = \{xt, x, t, 1\}$, respectively. This is the complete set of local conservation laws of **U** (1.1) *for an arbitrary wave speed $c(u)$, resulting from multipliers of the form $\Lambda_i(x,t,\tilde{u})$.*

(II). Potential systems following directly from the nonlinear wave equation U.

Each of the conservation laws (2.8)–(2.11) yields a nonlocally related potential system for the nonlinear wave equation **U** (1.1). The potential system **UV** (2.3) follows from the conservation law (2.11). From conservation laws (2.8)–(2.10), we obtain potential systems

$$\mathbf{UA}\{x,t,u,a\} = 0: \quad \begin{cases} a_x - (x[tu_t - u]) = 0, \\ a_t - (t[xc^2(u)u_x - \int c^2(u)\,du]) = 0; \end{cases} \tag{2.12}$$

$$\mathbf{UB}\{x,t,u,b\} = 0: \quad \begin{cases} b_x - xu_t = 0, \\ b_t - (xc^2(u)u_x - \int c^2(u)\,du) = 0; \end{cases} \tag{2.13}$$

$$\mathbf{UW}\{x,t,u,w\} = 0: \quad \begin{cases} w_x - (tu_t - u) = 0, \\ w_t - tc^2(u)u_x = 0. \end{cases} \tag{2.14}$$

Remark 1. No nonlocally related subsystems result from potential systems **UA**, **UB**, **UV**, **UW**.

Combining these four potential systems in couplets, triplets, and a quadruplet, one obtains a tree of nonlocally related systems for the given nonlinear wave equation **U** (1.1) (see [3]).

Therefore we have proved the following theorem.

Theorem 3. *For the nonlinear wave equation* **U** (1.1)*, for an* arbitrary *form of $c(u)$, the set of locally inequivalent potential systems arising from multipliers depending on x, t and u, is exhausted by the following systems*:

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- *four potential systems* (2.3), (2.14), (2.12) *and* (2.13) *involving single potentials*;
- *six couplets* (2.3, 2.14), (2.3, 2.12), (2.3, 2.13), (2.14, 2.12), (2.14, 2.13) *and* (2.12, 2.13) *involving pairs of potentials*;
- *four triplets* (2.3, 2.14, 2.12), (2.3, 2.14, 2.13), (2.3, 2.12, 2.13) *and* (2.14, 2.12, 2.13) *for combinations involving three potentials*;
- *one quadruplet* (2.3, 2.14, 2.12, 2.13) *involving all four potentials.*

(III). Nonlocally related systems following from the potential system XT (2.5).

The system **XT** (2.5) yields two nonlocally related subsystems

$$\mathbf{T}\{u,v,t\} \equiv \mathbf{L}\{u,v,t\} = 0: \quad t_{vv} - c^{-2}(u)t_{uu} = 0; \tag{2.15}$$

$$\mathbf{X}\{u,v,x\} = 0: \quad x_{vv} - \left(c^{-2}(u)x_u\right)_u = 0. \tag{2.16}$$

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Remark 2. It is easy to show that the linear wave equations **L** (2.15) and **X** (2.16) are related by a point transformation. Indeed, if $x(u,v)$ satisfies (2.16), then after the change of variables $u \to u_1$, $v \to v$, $x \to t$, where $u_1(u)$ satisfies

$$\frac{du_1(u)}{du} = c^2(u),$$

the function $t(u_1, v)$ satisfies the equation $t_{vv} - c^2(u_1)t_{u_1u_1} = 0$, which is of the type (2.15). Note that the change of variables changes the wave speed: $c(u) \to c^{-1}(u(u_1))$. Therefore *the symmetry classifications of Eqs.* (2.15) *and* (2.16) *are equivalent*. However *the relation between the two systems is nonlocal*.

Remark 3. Note that the linear PDE (2.15) is the adjoint of (2.16).

We now construct local conservation laws of the linear wave equation **L** (2.15) to obtain further nonlocally related PDE systems. Looking for multipliers of the form $\Lambda(u,v,\tilde{t}) = c^2(u)f(u,v,\tilde{t})$, we find that $f(u,v,\tilde{t})$ is any function that satisfies the linear equation

$$f_{uu}(u,v,\tilde{t}) = c^2(u)f_{vv}(u,v,\tilde{t}). \tag{2.17}$$

Indeed, for every wave speed $c(u)$, the linear equation (2.15) has an infinite number of conservation laws, with multipliers defined by (2.17). However it is easy to prove that *the only multipliers that hold for all wave speeds* arise from $\{f_1, f_2, f_3, f_4\} = \{1, u, v, uv\}$. These multipliers are $\{\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4\} = \{c^2(u), uc^2(u), vc^2(u), uvc^2(u)\}$. They yield conservation laws

$$D_u(t_u) - D_v\left(c^2(u)t_v\right) = 0, \tag{2.18}$$

$$D_u(ut_u - t) - D_v\left(uc^2(u)t_v\right) = 0, \tag{2.19}$$

$$D_u(vt_u) - D_v\left(c^2(u)(vt_v - t)\right) = 0, \tag{2.20}$$

$$D_u\left(v[ut_u - t]\right) - D_v\left(uc^2(u)[vt_v - t]\right) = 0, \tag{2.21}$$

which are *nonlocal conservation laws* of the given nonlinear wave equation **U** (1.1).

Conservation law (2.18) yields the previously known potential system **XT** (2.5). Conservation laws (2.19)–(2.21) respectively yield potential systems

$$\mathbf{TP}\{u,v,t,p\}=0: \quad \begin{cases} p_v-(ut_u-t)=0, \\ p_u-\big(uc^2(u)t_v\big)=0; \end{cases} \tag{2.22}$$

$$\mathbf{TQ}\{u,v,t,q\}=0: \quad \begin{cases} q_v-(vt_u)=0, \\ q_u+\big(c^2(u)(t-vt_v)\big)=0 \end{cases} \tag{2.23}$$

and

$$\mathbf{TR}\{u,v,t,r\}=0: \quad \begin{cases} r_v-\big(v[ut_u-t]\big)=0, \\ r_u-\big(uc^2(u)[vt_v-t]\big)=0. \end{cases} \tag{2.24}$$

Remark 4. No nonlocally related subsystems follow from potential systems **TP**, **TQ**, and **TR**.

Combining potential systems **XT**, **TP**, **TQ** and **TR** in couplets, triplets, and a quadruplet, we extend the tree of nonlocally related systems for the nonlinear wave equation **U** (1.1):

Theorem 4. *For an* arbitrary *form of* $c(u)$*, the linear wave equation* **L** (2.15) *is nonlocally related to the following systems*:

- *four potential systems* (2.5), (2.22), (2.23) *and* (2.24) *involving single potentials*;
- *six couplets* (2.5, 2.22), (2.5, 2.23), (2.5, 2.24), (2.22, 2.23), (2.22, 2.24) *and* (2.23, 2.24) *involving pairs of potentials*;
- *four triplets* (2.5, 2.22, 2.23), (2.5, 2.22, 2.24), (2.5, 2.23, 2.24) *and* (2.22, 2.23, 2.24) *for combinations involving three potentials*;
- *one quadruplet* (2.5, 2.22, 2.23, 2.24) *involving all four potentials*;
- *the subsystem* (2.16).

Remark 5. The list of potential systems of the linear wave equation **L** given in Theorem 4 is not exhaustive, since one may look for local conservation laws of system **L** arising for *specific forms* of $c(u)$, as well as nonlocal conservation laws, and thereby construct further potential

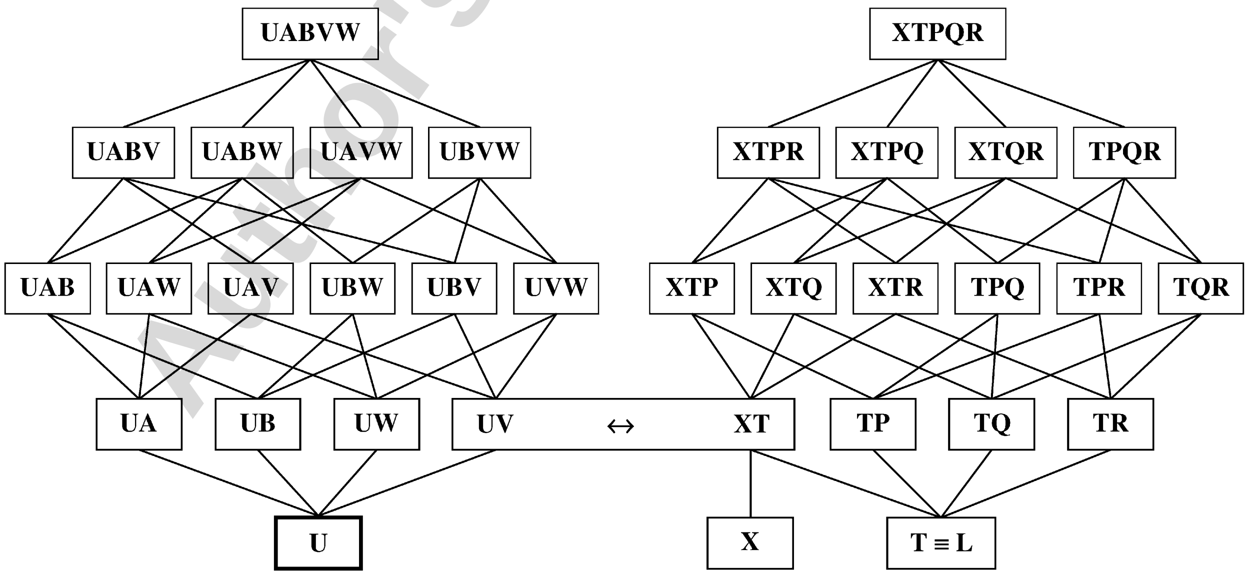

Fig. 1. A tree $\mathcal{T}$ of nonlocally related potential systems and subsystems for the nonlinear wave equation **U** (1.1). (Arrows within a box show local relations between the systems; lines between boxes denote nonlocal relations.)

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systems [2,3]. However in this paper we limit ourselves to considering only potential systems of the linear wave equation **L** following from local conservation laws valid for all wave speeds $c(u)$, i.e. the potential systems listed in Theorem 4.

The tree $\mathcal{T}$ of the above-listed nonlocally related potential systems and subsystems for the nonlinear wave equation **U** (1.1) for *arbitrary* $c(u)$ is shown in Fig. 1.

3. Nonlocal symmetries of the nonlinear wave equation

In this section we determine nonlocal symmetries of the nonlinear wave equation **U** (1.1) that arise as point symmetries of its nonlocally related systems found in Section 2.2. We only consider potential systems with one potential variable: **UA**, **UB**, **UV** ≡ **XT**, **UW**, **TP**, **TQ** and **TR**, and the nonlocally related subsystems **L** and **X**. In particular, we demonstrate that all of these systems, except for **UA**, yield new nonlocal symmetries of **U** (1.1) for specific classes of wave speeds $c(u)$.

For each of the nine nonlocally related systems, we determine its group of admitted equivalence transformations. For each family of wave speeds $c(u)$ related by equivalence transformations, symmetry classification results are given for the simplest form of $c(u)$. Symmetries for wave speeds related to a given $c(u)$ by an by equivalence transformation can be obtained directly by applying corresponding equivalence transformations to symmetry generators.

3.1. Point symmetries of the nonlinear wave equation

We start by reviewing previously known point symmetry analysis results for the nonlinear wave equation **U** (1.1). The group of equivalence transformations admitted by **U** is given by

$$x = a_1 x^* + a_4, \quad t = a_2 t^* + a_5, \quad u = a_3 u^* + a_6, \quad c(u) = a_1 a_2^{-1} c^*(u^*), \tag{3.25}$$

where $a_1, \ldots, a_7$ are arbitrary constants, and $a_1 a_2 a_3 \neq 0$. Ames et al. [1] showed that the symmetry group classification of the nonlinear wave equation **U** (1.1) is as shown in Table 1 (modulo equivalence transformations (3.25)).

In Section 2.1, we showed that for every $c(u)$ the nonlinear wave equation **U** (1.1) is equivalent to and nonlocally related to the linear wave equation **L** (1.2). We note that for $c(u) = u^{-2}$, the linear wave equation can be further simplified since it can be invertibly mapped into the *constant-coefficient wave equation* [9].

Table 1
Point symmetries of the nonlinear wave equation **U** (1.1)

$c(u)$	Symmetries
Arbitrary	$X_1 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$, $X_2 = \frac{\partial}{\partial t}$, $X_3 = \frac{\partial}{\partial x}$.
u^C	X_1, X_2, X_3, and $X_4 = Cx\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}$.
u^{-2}	X_1, X_2, X_3, X_4 $(C = -2)$, and $X_5 = t^2\frac{\partial}{\partial t} + tu\frac{\partial}{\partial u}$.
$u^{-2/3}$	X_1, X_2, X_3, X_4 $(C = -\frac{2}{3})$, and $X_6 = x^2\frac{\partial}{\partial x} - 3xu\frac{\partial}{\partial u}$.
e^u	$X_1, X_2, X_3, X_7 = x\frac{\partial}{\partial x} + \frac{\partial}{\partial u}$.

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3.2. Nonlocal symmetries

Nonlocal symmetries of the nonlinear wave equation **U** (1.1) are sought through seeking point symmetries of PDE systems nonlocally related to **U** (i.e. systems in the tree $\mathcal{T}$; see Fig. 1), and then isolating symmetries that do not arise in the point symmetry analysis of **U**. In particular, a nonlocal symmetry is obtained if infinitesimal symmetry components of local variables (x, t, u) depend on *nonlocal variables*.

Point symmetries of the linear wave equation **L** (1.2) and its potential system **XT** (2.5) were classified in [4]. Point symmetries of **X** (2.16) and system **TP** (2.22) were incompletely classified in [6].

Here we study the potential systems **UA**, **UB**, **UV** ≡ **XT**, **UW**, **TP**, **TQ** and **TR**, and subsystems **L** and **X**, and classify wave speeds $c(u)$ that yield nonlocal symmetries of **U** (1.1). The results for system **XT** (2.5) are adapted from [4]; the results for systems **TP** (2.22) and **X** (2.16) include adaptations from [6]; the results for system **TQ** (2.23) are partially adapted from [6]. The systems **UA**, **UB**, **UW** and **TR** are considered here for the first time.

Analyzing the point symmetries of the nine above-mentioned nonlocally related systems, we summarize the situations in which nonlocal symmetries of **U** (1.1) arise in Table 2. The results are given modulo the equivalence transformations (3.25).

We now specify the nonlocal symmetries for the cases listed in Table 2.

3.2.1. *The potential system* **UB** (2.13)

The potential system **UB** (2.13) admits the group of equivalence transformations

$$\begin{aligned} &x = a_1x^*, \quad t = a_2t^* + a_4, \quad u = a_3u^* + a_5, \\ &b = a_1^2a_2^{-1}a_3b^* + a_6 - a_2a_7t^*, \quad F(u) = a_1^2a_2^{-2}a_3F^*(u^*) + a_7, \end{aligned} \tag{3.26}$$

where $F(u) = \int c^2(u)\,du$; $a_1, \ldots, a_7$ are arbitrary constants with $a_1a_2a_3 \neq 0$.

For an arbitrary wave speed $c(u)$, the system **UB** (2.13) admits a three-dimensional symmetry algebra

$$Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = \frac{\partial}{\partial b}, \quad Y_3 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + b\frac{\partial}{\partial b}.$$

These point symmetries project onto point symmetries admitted by the nonlinear wave equation **U**.

If the wave speed $c(u)$ satisfies the ODE

$$\frac{F''(u)}{(F'(u))^2} = \frac{4F(u) + C_1}{(F(u) + C_2)^2 + C_3}, \tag{3.27}$$

where $F(u) = \int c^2(u)\,du$ and C_1, C_2, C_3 are arbitrary constants, then the potential system **UB** admits an additional point symmetry

$$Y_4 = \left(F(u) + \frac{C_1}{2}\right)x\frac{\partial}{\partial x} + b\frac{\partial}{\partial t} + \frac{(F(u) + C_2)^2 + C_3}{F'(u)}\frac{\partial}{\partial u} + \left(2C_2b - \left(C_2^2 + C_3\right)t\right)\frac{\partial}{\partial b},$$

which is a nonlocal symmetry of **U** that is new since it is not obtainable from known results in [4,6].

For $c(u) = u^{-2/3}$, the potential system **UB** admits an infinite number of point symmetries. This suggests that it is linearizable by a point transformation. Indeed, the following lemma holds.

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Table 2
Cases in which nonlocal symmetries of the nonlinear wave equation **U** (1.1) arise

System	Nonlocal variable(s)	Condition on $c(u)$	Symmetries; remarks
UA (2.12)	a	No special cases	Nonlocal symmetries do not arise.
UB (2.13)	b	$c(u)=u^{-2/3}$	Linearizable by a point transformation.
		$\frac{F''(u)}{(F'(u))^2}=\frac{4F(u)+C_1}{(F(u)+C_2)^2+C_3}$ $(F(u)=\int c^2(u)\,du,\ C_1, C_2, C_3=\text{const})$	One nonlocal symmetry.
UW (2.14)	w	$c(u)=u^{-2}$	Linearizable by a point transformation.
		$\frac{c'(u)}{c(u)}=-\frac{2u+C_1}{u^2+C_2}$ $(C_1, C_2=\text{const})$	One nonlocal symmetry.
XT (2.5) or **UV** (2.3)	v	$[\frac{c'(u)}{c^3(u)}(\frac{c(u)}{c'(u)})'']'=0$	One or two nonlocal symmetries; adapted from [4].
TP (2.22)	v, p	$\frac{-(2uc^2+u^2cc')c'''+2u^2c(c'')^2}{c^3(uc'+2c)^2}$ $+\frac{-(4c^2+u^2(c')^2-8ucc')c''+6(c-uc')(c')^2}{c^3(uc'+2c)^2}=\lambda^2,$ $\lambda=\text{const}$	One or two nonlocal symmetries; partially adapted from [6].
		$c(u)=u^{-2}$	Infinite number of nonlocal symmetries; there exists a point mapping into a system with constant coefficients.
TQ (2.23)	v, q	$c(u)=u^{-2/3}$; $c(u)=u^{-2}$	Two nonlocal symmetries; partially adapted from [6].
TR (2.24)	v, r	$\frac{ucc''+(c-uc')c'}{(uc'+2c)^2}=\gamma^2=\text{const}$	Two nonlocal symmetries.
L (2.15)	v	$(\alpha'+H\alpha)'=\sigma^2\alpha c^2(u)$, $\sigma=\text{const}$ $(H=c'(u)/c(u), \alpha^2=(H^2-2H')^{-1})$	One or two nonlocal symmetries; adapted from [4].
		$c(u)=u^{-2}$	Infinite number of nonlocal symmetries; there exists an invertible mapping into a system with constant coefficients [9].
X (2.16)	v	$\frac{(-2cc''+5(c')^2)c^2c''''+3c^3(c''')^2+16c^2(c'')^3}{c^3(2cc''-5(c')^2)^2}$ $+\frac{-24c^2c'c''c'''+12c(c'c'')^2-10(c')^4c''}{c^3(2cc''-5(c')^2)^2}=\sigma^2,$ $\sigma=\text{const}$	One or two nonlocal symmetries; partially adapted from [6].

Lemma 1. *For $c(u)=u^{-2/3}$, the nonlinear PDE system* **UB** *is linearizable by a point transformation.*

Proof. We introduce new variables $s=x^{-1}$, $\beta=x^3u$. For $c(u)=u^{-2/3}$, the system **UB** takes the form

$$\mathbf{UB}\{s,t,u,\beta\}=0:\quad \begin{cases} b_s+\beta_t=0,\\ b_t+\beta^{-4/3}\beta_s=0, \end{cases} \tag{3.28}$$

which is linearizable by a point hodograph transformation [9]. □

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3.2.2. The potential system **UW** (2.14)

The group of equivalence transformations admitted by the potential system **UW** (2.14) consists of transformations

$$x = a_1 x^* + a_4, \quad t = a_2 t^*, \quad u = a_3 u^* + a_6 t^* + a_7,$$
$$w = a_1 a_3 w^* - a_1 a_7 x^* + a_5, \quad c(u) = a_1 a_2^{-1} c^*\big(u^*\big), \tag{3.29}$$

where $a_1, \dots, a_7$ are arbitrary constants, $a_1 a_2 a_3 \neq 0$, as well as

$$x = x^* - a_8 w, \quad t = \frac{t^*}{1 + a_8 u^*}, \quad u = \frac{u^*}{1 + a_8 u^*}, \quad w = w^*,$$
$$c(u) = \big(1 + a_8 u^*\big)^2 c^*\big(u^*\big). \tag{3.30}$$

In particular, the equivalence transformation (3.30) shows that the system **UW** with wave speed $c(u)$ is equivalent to the same system with wave speed

$$c(u) = (1 + a_8 u)^{-2} c\left(\frac{u}{1 + a_8 u}\right) \tag{3.31}$$

for an arbitrary constant a_8.

For an arbitrary wave speed $c(u)$, the potential system **UW** (2.14) admits a three-dimensional symmetry algebra

$$Z_1 = \frac{\partial}{\partial x}, \quad Z_2 = \frac{\partial}{\partial w}, \quad Z_3 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + w\frac{\partial}{\partial w}.$$

These point symmetries project onto point symmetries admitted by **U**.

If the wave speed $c(u)$ satisfies the ODE

$$\frac{c'(u)}{c(u)} = -\frac{2u + C_1}{u^2 + C_2}, \tag{3.32}$$

where C_1, C_2 are arbitrary constants, then the potential system **UW** admits an additional point symmetry

$$Z_4 = w\frac{\partial}{\partial x} + (u + C_1)t\frac{\partial}{\partial t} + \big(u^2 + C_2\big)\frac{\partial}{\partial u} - C_2 x\frac{\partial}{\partial w},$$

which is a nonlocal symmetry of **U** that is new since it is not obtainable from known results in [4,6].

The general solution of (3.32) is found as follows:

$$C_2 = \omega^2 > 0: \quad c(u) = \frac{c_0}{u^2 + \omega^2} \exp\left\{-\frac{C_1}{\omega} \tan^{-1}\frac{u}{\omega}\right\};$$
$$C_2 = -\omega^2 < 0: \quad c(u) = \frac{c_0}{u^2 - \omega^2} \left|\frac{u + \omega}{u - \omega}\right|^{\frac{C_1}{2\omega}};$$
$$C_2 = 0: \quad c(u) = \frac{c_0}{u^2} \exp\frac{C_1}{u}. \tag{3.33}$$

Here c_0 is an arbitrary constant of integration.

For $c(u) = u^{-2}$ the system **UW** admits an infinite number of point symmetries. We show that here **UW** is linearizable by a point transformation.

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Lemma 2. *For $c(u) = u^{-2}$, the nonlinear PDE system* **UW** *is linearizable by a point transformation.*

Proof. Introduce new variables $z = t^{-1}$, $\alpha = u/t$. Then system **UW** (2.14) takes the form

$$\mathbf{UW}\{x, z, \alpha, w\} = 0: \quad \begin{cases} w_x + \alpha_z = 0, \\ w_z + \alpha^{-4}\alpha_x = 0, \end{cases} \tag{3.34}$$

which is linearizable by a point hodograph transformation. □

3.2.3. The potential system **XT** (2.5)

The equivalence transformations of the system **XT** (2.5) are given by

$$\begin{aligned} &u = a_1u^* + a_4 + a_8t, \quad v = a_2v^* + a_5 + a_8x, \\ &x = a_3x^* + a_6 + a_9v, \quad t = a_1a_2^{-1}a_3t^* + a_7 + a_9u, \quad c(u) = a_1^{-1}a_2c^*\big(u^*\big), \end{aligned} \tag{3.35}$$

where $a_1, \ldots, a_9$ are arbitrary constants, with $a_1a_2a_3 \neq 0$.

Point symmetries of **XT** were classified in [4]. For an arbitrary wave speed $c(u)$, the system **XT** admits a three-dimensional symmetry algebra which corresponds to the point symmetries admitted by the nonlinear wave equation **U** (1.1) that are exhibited in Table 1.

Additional symmetries arise when $c(u)$ satisfies

$$\frac{c'(u)}{c^3(u)}\left(\frac{c(u)}{c'(u)}\right)'' = \lambda^2 = \text{const.} \tag{3.36}$$

Point symmetries of system **XT** are summarized in Table 3. For several classes of wave speeds $c(u)$ satisfying (3.36) there are point symmetries admitted by system **XT** (2.5) that yield nonlocal symmetries of **U** (1.1).

Point symmetries W_6, W_7, W_9, W_{10}, W_{11} of the system **XT** correspond to nonlocal symmetries of the nonlinear wave equation **U** (1.1).

Table 3
Point symmetries of the potential system **XT** (2.5)

$c(u)$	Admitted point symmetries
Arbitrary	$W_1 = \frac{\partial}{\partial t}$, $W_2 = \frac{\partial}{\partial x}$, $W_3 = \frac{\partial}{\partial v}$, $W_4 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$.
u^C $(C \neq 0, -1)$	$W_1, W_2, W_3, W_4, W_5 = Ct\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - (1+C)v\frac{\partial}{\partial v}$,
	$W_6 = -((2C+1)tv + xu)\frac{\partial}{\partial t} - (tu^{1+2C} + xv)\frac{\partial}{\partial x} + 2uv\frac{\partial}{\partial u} + [(1+C)v^2 + \frac{u^{2+2C}}{1+C}]\frac{\partial}{\partial v}$.
u^{-1}	W_1, W_2, W_3, W_4, W_5 $(C = -1)$,
	$W_7 = (tv - xu)\frac{\partial}{\partial t} - (tu^{-1} + xv)\frac{\partial}{\partial x} + 2uv\frac{\partial}{\partial u} + 2\log u\frac{\partial}{\partial v}$.
e^u	$W_1, W_2, W_3, W_4, W_8 = x\frac{\partial}{\partial x} + \frac{\partial}{\partial u} + v\frac{\partial}{\partial v}$,
	$W_9 = -(2vt + x)\frac{\partial}{\partial t} - 2e^ut\frac{\partial}{\partial x} + 4v\frac{\partial}{\partial u} + (4e^u + v^2)\frac{\partial}{\partial v}$.
$c(u)$ satisfies (a), (b) or (c):	W_1, W_2, W_3, W_4,
(a) $c' = c^2\nu^{-1}\sinh(\nu\log c)$	$W_{10,11} = e^v\{[(2+\Gamma')t \pm \Gamma x]\frac{\partial}{\partial t} + [\Gamma'x \pm c^2\Gamma t]\frac{\partial}{\partial x} - 2\Gamma\frac{\partial}{\partial u} \mp 2(\Gamma'+1)\frac{\partial}{\partial v}\}$,
(b) $c' = c^2\nu^{-1}\sin(\nu\log c)$	where $\Gamma = c/c'$.
(c) $c' = c^2\nu^{-1}\cosh(\nu\log c)$	

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3.2.4. The potential system **TP** (2.22)

The group of equivalence transformations admitted by **TP** (2.22) consists of transformations

$$u = a_1 u^*, \quad v = a_2 v^* + a_4,$$
$$p = a_3 p^* + a_5 - a_2 a_6 v^*, \quad t = a_2^{-1} a_3 t^* + a_6 + a_7 u^*, \quad c(u) = a_1^{-1} a_2 c^*(u^*), \tag{3.37}$$

and

$$u = \frac{u^*}{1 + a_8 u^*}, \quad v = v^*, \quad p = p^*, \quad t = \frac{t^*}{1 + a_8 u^*}, \quad c(u) = (1 + a_8 u^*)^2 c^*(u^*). \tag{3.38}$$

where $a_1, \dots, a_8$ are arbitrary constants, $a_1 a_2 a_3 \neq 0$. It follows that for the system **TP**, wave speeds within the family (3.31) yield equivalent **TP** systems.

The symmetry classification of system **TP** is as follows.

(i) For an arbitrary wave speed $c(u)$, the system **TP** (2.22) admits a three-dimensional symmetry algebra

$$L_1 = \frac{\partial}{\partial v}, \quad L_2 = t\frac{\partial}{\partial t} + p\frac{\partial}{\partial p}, \quad L_3 = \frac{\partial}{\partial p}.$$

These symmetries project onto point symmetries admitted by **U**.

(ii) For $c(u) = u^{-2}$, the PDE system **TP** (2.22) admits an infinite number of point symmetries that are related to symmetries of system **XT** (2.5) with $c(u) = \text{const}$. Indeed, for $c(u) = u^{-2}$, the system **TP** is mapped by the transformation $y = -1/u$, $\gamma = t/u$ into a system with constant coefficients

$$\mathbf{TP}\{y, v, \gamma, p\} = 0: \quad \begin{cases} p_v - \gamma_y = 0, \\ p_y - \gamma_v = 0. \end{cases}$$

Note that for $c(u) = (u + B)^{-2}$, $B \neq 0$, system **TP** does not admit an infinite number of point symmetries since system **TP** is not invariant under translations in u.

For $c(u) \neq u^{-2}$, and $c(u)$ satisfying

$$\frac{-(2uc^2 + u^2 cc')c''' + 2u^2 c(c'')^2 - (4c^2 + u^2(c')^2 - 8ucc')c'' + 6(c - uc')(c')^2}{c^3(uc' + 2c)^2} = \lambda^2, \tag{3.39}$$

with λ a real or imaginary constant, system **TP** (2.22) admits additional point symmetries [6].

(iii) When $\lambda \neq 0$, additional symmetries are given by

$$\begin{aligned} L_{4,5} = e^{\pm\lambda v}\Bigg\{&\Bigg[\pm\frac{\lambda^2 u^2 c}{2(2c + uc')} t \\ &- \left(\lambda\frac{c + uc'}{2c + uc'} - \frac{u^2(cc'' - 3(c')^2) - 4ucc' - 2c^2}{(2c + uc')^2}\right)p\Bigg]\frac{\partial}{\partial p} \\ &\pm\left[\frac{\lambda^2 c}{2(2c + uc')}p + \left(\frac{u^2(cc'' - 3(c')^2) - 4ucc' - 2c^2}{2(2c + uc')^2}\right)t\right]\frac{\partial}{\partial t} \\ &- \frac{\lambda uc}{2c + uc'}\frac{\partial}{\partial u} \pm \left[\frac{u^2(cc'' - 2(c')^2) - 2ucc' - 2c^2}{(2c + uc')^2}\right]\frac{\partial}{\partial v}\Bigg\} \end{aligned} \tag{3.40}$$

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which, due to the factor $e^{\pm\lambda v}$, yield *nonlocal symmetries* of **U** (1.1).

(iv) When $\lambda = 0$, the general solution of Eq. (3.39) consists of three classes

$$c(u) = Au^C (u + B)^{-2-C}; \tag{3.41}$$

$$c(u) = Au^C; \tag{3.42}$$

$$c(u) = Au^{-2} e^{B/u}. \tag{3.43}$$

Here A, B, C are nonzero constants, $C \neq -2$. This case was not considered in [6].

From equivalence transformations (3.37), (3.38) (see also (3.31)) it follows that systems **TP** with wave speeds (3.41) are equivalent to systems **TP** with wave speeds (3.42). Therefore we consider only the nonequivalent cases (3.42), (3.43) (modulo equivalence transformations (3.37), (3.38)).

(iv)[a] For wave speeds $c(u) = u^C$, with $C \neq -1$, system **TP** has two additional symmetries

$$L_6 = Ct\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - (1 + C)v\frac{\partial}{\partial v} - p\frac{\partial}{\partial p},$$

$$L_7 = -\big((2C + 1)tv + p\big)\frac{\partial}{\partial t} + 2uv\frac{\partial}{\partial u} + \left[(1 + C)v^2 + \frac{u^{2+2C}}{1 + C}\right]\frac{\partial}{\partial v} + \big(tu^{2+2C} - vp\big)\frac{\partial}{\partial p}.$$

We note that the generator L_7 is nonlocal for **U** but local for system **XT**; symmetries L_6 and L_7 correspond to W_5 and W_6, respectively, in Table 3.

(iv)[b] For $c(u) = u^{-1}$, system **TP** has two additional symmetries

$$L_6 \quad (C = -1),$$

$$L_8 = (tv - p)\frac{\partial}{\partial t} + 2uv\frac{\partial}{\partial u} + 2\log u\frac{\partial}{\partial v} - (t - pv)\frac{\partial}{\partial p}.$$

Symmetry L_8 is nonlocal for **U** but local for system **XT**. These symmetries correspond to W_5 ($C = -1$) and W_7, respectively, in Table 3.

(iv)[c] For $c(u) = u^{-2}e^{1/u}$, the system **TP** has two additional symmetries

$$L_9 = \big(pu - 2tv(u + 1)\big)\frac{\partial}{\partial t} - 2u^2 v\frac{\partial}{\partial u} + \big(u^2 + e^{2/u}\big)\frac{\partial}{\partial v} + t\frac{e^{2/u}}{u}\frac{\partial}{\partial p},$$

$$L_{10} = t(u + 1)\frac{\partial}{\partial t} + u^2\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}.$$

Both symmetries L_9 and L_{10} are nonlocal for **U** as well as system **XT**.

3.2.5. The potential system **TQ** (2.23)

The group of equivalence transformations admitted by **TQ** (2.23) consists of transformations

$$u = a_1 u^* + a_4, \quad v = a_2 v^*, \quad q = a_3 q^* + a_5 + \frac{a_2^2 a_7}{3} v^{*3},$$

$$t = a_1 a_2^{-2} a_3 t^* + a_2 a_6 v^* + a_1 a_7 u^* v^*, \quad c(u) = a_1^{-1} a_2 c^*\big(u^*\big), \tag{3.44}$$

where $a_1, \ldots, a_8$ are arbitrary constants, $a_1 a_2 a_3 \neq 0$.

The point symmetry classification of the potential system **TQ** (2.23) is given in Table 4.

For $c(u) = u^{-2}$ and $c(u) = u^{-2/3}$, the system **TQ** (2.23) admits a five-dimensional symmetry algebra, with the extra symmetries (M_5, M_6) and (M_6, M_7), respectively, yielding *nonlocal symmetries* of the given wave equation **U**.

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Table 4
Point symmetries of the potential system **TQ** (2.23)

$c(u)$	Admitted point symmetries
Arbitrary	$M_1 = \frac{\partial}{\partial q}$, $M_2 = t\frac{\partial}{\partial t} + q\frac{\partial}{\partial q}$.
u^C	$M_1, M_2, M_3 = (2C+1)t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - (C+1)v\frac{\partial}{\partial v}$.
e^u	$M_1, M_2, M_4 = 2t\frac{\partial}{\partial t} - \frac{\partial}{\partial u} - v\frac{\partial}{\partial v}$.
u^{-2}	$M_1, M_2, M_3, M_5 = \frac{u^2}{u^2v^2-1}[t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}]$,
	$M_6 = \frac{1}{u^2}[(4u^3q - 5tv^2u^2 - 3t)\frac{\partial}{\partial t} - (3u^2v^2+1)u\frac{\partial}{\partial u} + (u^2v^2+3)v\frac{\partial}{\partial v} + \frac{2}{u}(2tv^2 + (u^2v^2+1)uq)\frac{\partial}{\partial q}]$.
$u^{-2/3}$	M_1, M_2, M_3, M_7, M_8 (For determining equations yielding M_7, M_8 see Appendix A.)

3.2.6. The potential system **TR** (2.24)

The group of equivalence transformations admitted by **TR** (2.24) consists of transformations

$$u = a_1u^*, \quad v = a_2v^*, \quad r = a_3r^* + a_4 - a_2^2a_6\frac{(v^*)^3}{3},$$
$$t = a_2^{-2}a_3t^* + a_5u^*v^* + a_6v^*, \quad c(u) = a_1^{-1}a_2c^*(u^*), \tag{3.45}$$

and

$$u = \frac{u^*}{1+a_7u^*}, \quad v = v^*, \quad p = p^*, \quad t = \frac{t^*}{1+a_7u^*}, \quad c(u) = (1+a_7u^*)^2c^*(u^*). \tag{3.46}$$

where $a_1, \dots, a_7$ are arbitrary constants, $a_1a_2a_3 \neq 0$.

The equivalence transformation (3.46) shows that for the system **TR**, wave speeds $c(u)$ are equivalent within the family given by (3.31). Hence the wave speeds

$$c(u) = u^C \quad \text{and} \quad c(u) = u^C(u+B)^{-2-C} \quad (C \neq -2), \tag{3.47}$$

are equivalent, and in particular (using (3.45)),

1. $c(u) = u^{-1}$ and $c(u) = u^{-1}(Au+B)^{-1}$;
2. $c(u) = u^{-4/3}$ and $c(u) = u^{-4/3}(Au+B)^{-2/3}$;
3. $c(u) = 1$ and $c(u) = (Au+B)^{-2}$;

Here A, B are *nonzero constants*. Note that $c(u) = u^{-2}$ is invariant under the equivalence transformation (3.47).

The point symmetry classification of the potential system **TR** (2.24) (modulo the equivalence transformations (3.45)–(3.47)) is given in Table 5.

An additional symmetry of system **TR** arises for $c(u)$ satisfying

$$\frac{ucc'' + (c - uc')c'}{(uc' + 2c)^2} = \gamma^2 = \text{const.} \tag{3.48}$$

The general solution of (3.48) for $\gamma \neq 0$ (modulo the equivalence transformations (3.45), (3.46)) consists of two families: (a) $c(u) = u^C$ ($C = \text{const}$) and (b) $c(u) = u^{-2}e^{1/u}$.

For $c(u)$ satisfying (3.48) with $\gamma = 0$, two more symmetries arise for system **TR**. The general solution of (3.48) for $\gamma \neq 0$ (modulo the equivalence transformations (3.45), (3.46)) in this case is given by $c(u) = u^{-4/3}$.

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Table 5
Point symmetries of the potential system **TR** (2.24)

$c(u)$	Admitted point symmetries
Arbitrary	$N_1=\frac{\partial}{\partial r}$, $N_2=t\frac{\partial}{\partial t}+r\frac{\partial}{\partial r}$.
u^C, $C\neq -2$	$N_1, N_2, N_3=2(C+1)t\frac{\partial}{\partial t}-u\frac{\partial}{\partial u}-(C+1)v\frac{\partial}{\partial v}$.
$u^{-2}e^{1/u}$	$N_1, N_2, N_4=(u+1)t\frac{\partial}{\partial t}+u^2\frac{\partial}{\partial u}-v\frac{\partial}{\partial v}-r\frac{\partial}{\partial r}$.
$u^{-4/3}$	N_1, N_2, N_5, N_6 (For determining equations for N_5, N_6 see Appendix B.)
1	N_1, N_2, N_3 $(C=0)$, $N_7=\frac{1}{u^2-v^2}(u\frac{\partial}{\partial u}-v\frac{\partial}{\partial v})$,
	$N_8=2[t(u^2+v^2)+2r]\frac{\partial}{\partial t}-u(u^2+3v^2)\frac{\partial}{\partial u}-v(3u^2+v^2)\frac{\partial}{\partial v}+2[2tu^2v^2-r(u^2+v^2)]\frac{\partial}{\partial r}$.
u^{-2}	N_1, N_2, N_3 $(C=-2)$, $N_9=tu\frac{\partial}{\partial t}+u^2\frac{\partial}{\partial u}$,
	$N_{10}=\frac{1}{u}[(tu^2v^2+2t-u^2r)\frac{\partial}{\partial t}+2v\frac{\partial}{\partial v}+(tv^2+r)\frac{\partial}{\partial r}]-(1+u^2v^2)\frac{\partial}{\partial u}$.

Comparing Tables 1 and 5, we conclude that the point symmetries $N_4,\dots,N_{10}$ of system **TR** (2.24) yield *nonlocal symmetries* of **U** that are new since they are not obtainable from known results in [4,6].

3.2.7. The nonlocally related subsystem **L** (2.15)

The group of equivalence transformations admitted by the linear wave equation **L** (2.15) consists of transformations

$$u=a_1u^*+a_4,\quad v=a_2v^*+a_5,\quad t=a_3t^*+a_6+a_7u^*+a_8v^*+a_9u^*v^*,$$
$$c(u)=a_1^{-1}a_2c^*(u^*),\tag{3.49}$$

and

$$u=\frac{u^*}{1+a_{10}u^*},\quad v=v^*,\quad t=\frac{t^*}{1+a_{10}u^*},\quad c(u)=(1+a_{10}u^*)^2c^*(u^*),\tag{3.50}$$

where $a_1,\dots,a_{10}$ are arbitrary constants, $a_1a_2a_3\neq 0$.

The point symmetry classification of **L** (adapted from [4]), modulo the equivalence transformations (3.49), (3.50), is given in Table 6.

By comparison with the point symmetries of the nonlinear wave equation **U** (1.1) (see Section 3.1), one observes that operators K_4, K_5, K_7, K_8, K_{10}, K_{11} and K_{12} define *nonlocal symmetries* of the nonlinear wave equation **U** (1.1).

3.2.8. The nonlocally related subsystem **X** (2.16)

The group of equivalence transformations admitted by **X** consists of transformations

$$u=a_1u^*+a_4,\quad v=a_2v^*+a_5,\quad x=a_3x^*+a_6+a_7v^*,\quad c(u)=a_1^{-1}a_2c^*(u^*),\tag{3.51}$$

where $a_1,\dots,a_7$ are arbitrary constants, $a_1a_2a_3\neq 0$.

The point symmetry classification of **X** was partially obtained in [6].

For the wave speed $c(u)=u^{-2/3}$, the equation **X** admits an infinite number of point symmetries, which suggests that it can be mapped by a point transformation into a constant coefficient equation.

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Table 6
Point symmetries of the subsystem **L** (2.15)

$c(u)$	Symmetries
Arbitrary	$K_1 = t\frac{\partial}{\partial t}$, $K_2 = \frac{\partial}{\partial v}$, $K_3 = \frac{\partial}{\partial t}$.
u^{-2}	Infinite number of nonlocal symmetries; there exists a point mapping into a system with constant coefficients [4].
$[(Bu^2 + C)$ $\times \exp\{A\int (Bu^2+C)^{-1}\,du\}]^{-1}$ $(A, B, C = \text{const})$	$K_1, K_2, K_3, K_4 = \frac{1}{2}t(A+2Bu)\frac{\partial}{\partial t} + (Bu^2+C)\frac{\partial}{\partial u} - Av\frac{\partial}{\partial v}$, $K_5 = \frac{1}{2}t(A+2Bu)v\frac{\partial}{\partial t} + (Bu^2+C)v\frac{\partial}{\partial u}$ $+ [-\frac{1}{2}Av^2 + \int c^2(u)(Bu^2+C)\,du]\frac{\partial}{\partial v}$.
u^C, $C \neq 0, -1, -2$	$K_1, K_2, K_3, K_6 = u\frac{\partial}{\partial u} + (1+C)v\frac{\partial}{\partial v}$, $K_7 = -\frac{1}{2}Ctv\frac{\partial}{\partial t} + uv\frac{\partial}{\partial u} + [\frac{u^{2+2C}}{1+C} + \frac{1}{2}(1+C)v^2]\frac{\partial}{\partial v}$.
u^{-1}	K_1, K_2, K_3, K_6 $(C=-1)$, $K_8 = \frac{1}{2}tv\frac{\partial}{\partial t} + uv\frac{\partial}{\partial u} + (\log u)\frac{\partial}{\partial v}$.
e^u	$K_1, K_2, K_3, K_9 = \frac{\partial}{\partial u} + v\frac{\partial}{\partial v}$, $K_{10} = -\frac{1}{2}tv\frac{\partial}{\partial t} + v\frac{\partial}{\partial u} + \frac{1}{2}[e^{2u} + v^2]\frac{\partial}{\partial v}$.
$c(u)$ satisfies $(\alpha' + H\alpha)' = \sigma^2 c^2(u)\alpha$, where $\sigma = \text{const} \neq 0$, $H(u) = c'(u)/c(u)$, $\alpha^2(u) = (H^2(u) - 2H'(u))^{-1}$	$K_1, K_2, K_3, K_{11,12} = e^{\pm\sigma v}[-\frac{1}{2}t\alpha H\frac{\partial}{\partial t} + \alpha\frac{\partial}{\partial u} \pm \sigma^{-1}(\alpha' + H\alpha)\frac{\partial}{\partial v}]$.

Table 7
Point symmetries of the potential system **X** (2.16)

$c(u)$	Admitted point symmetries
Arbitrary	$J_1 = x\frac{\partial}{\partial x}$, $J_2 = \frac{\partial}{\partial v}$, $J_3 = \frac{\partial}{\partial x}$.
(3.52) $(\sigma \neq 0)$	$J_1, J_2, J_3, J_{4,5} = e^{\pm\sigma v}\{\frac{1}{2}xFH\frac{\partial}{\partial x} + F(v)\frac{\partial}{\partial u} \pm \sigma^{-1}[F' + FH]\frac{\partial}{\partial v}\}$.
(3.52) $(\sigma = 0)$	$J_1, J_2, J_3, J_6 = v\{\frac{1}{2}xFH\frac{\partial}{\partial x} + F\frac{\partial}{\partial u}\} + \{\frac{1}{2}Kv^2 + \int c^2F\,du\}\frac{\partial}{\partial v}$, $J_7 = \frac{1}{2}xFH\frac{\partial}{\partial x} + F\frac{\partial}{\partial u} + Kv\frac{\partial}{\partial v}$.
Particular case (a) for $\sigma = 0$: $c(u) = u^C$ $(C = \text{const})$	$J_6^{(a)} = C(C+1)xv\frac{\partial}{\partial x} + 2(C+1)uv\frac{\partial}{\partial u} + [u^{2C+2} + v^2(C+1)^2]\frac{\partial}{\partial v}$, $J_7^{(a)} = u\frac{\partial}{\partial u} + (C+1)v\frac{\partial}{\partial v}$.
Particular case (b) for $\sigma = 0$: $c(u) = e^u$	$J_6^{(b)} = xv\frac{\partial}{\partial x} + 2v\frac{\partial}{\partial u} + [e^{2u} + v^2]\frac{\partial}{\partial v}$, $J_7^{(b)} = \frac{\partial}{\partial u} + v\frac{\partial}{\partial v}$.

Additional symmetries of **X** arise when $c(u)$ satisfies

$$\frac{c^2(-2cc'' + 5(c')^2)c'''' + 3c^3(c''')^2 + 16c^2(c'')^3 - 24c^2c'c''c''' + 12c(c'c'')^2 - 10(c')^4c''}{c^3(2cc'' - 5(c')^2)^2} = \sigma^2 = \text{const}. \tag{3.52}$$

Symmetries of **X** (2.16) are summarized in Table 7. For several classes of wave speeds $c(u)$ there are point symmetries admitted by equation **X** that yield nonlocal symmetries of **U** (1.1). In Table 7, $F(u) = (3H^2(u) - 2H'(u))^{-1/2}$, $H(u) = c'(u)/c(u)$.

From symmetry commutator relations it is possible to show [6] that

$$(F' + HF)^2 - (\sigma cF)^2 = K^2 = \text{const},$$

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and hence for $\sigma = 0$, $F' + HF = K = \text{const}$.

From Table 7 we see that symmetries J_4, J_5 and J_6 yield *nonlocal symmetries* of **U**; J_7 yields a nonlocal symmetry of **U** except for the two listed particular cases.

4. Concluding remarks

In this paper, we have shown how the work of Bill Ames et al. in 1981 on classifying the point symmetries admitted by the nonlinear wave equation can be significantly extended to admitted nonlocal symmetries by consideration of admitted point symmetries of its nonlocally related systems. The starting point of our nonlocal symmetry analysis is the systematic construction of nonlocally related potential systems. The potential systems naturally arise from admitted local conservation laws. Furthermore, it is important to consider nonlocally related subsystems of such potential systems.

In this paper, we have only considered such nonlocally related systems that arise for an arbitrary wave speed. It is important to further consider the situation when classifying conservation laws with respect to specific classes of wave speeds, followed by consideration of the admitted point symmetries for the resulting additional nonlocally related systems. Moreover, we did not consider the situation of finding additional admitted nonlocal symmetries that result from determining admitted point symmetries of combined potential systems, i.e., the remaining 22 nonlocally related potential systems exhibited in Fig. 1.

Although a hodograph transformation linearizes one of the potential systems for any nonlinear wave equation, this does not really help one to find useful (nontrivial) admitted symmetries in order to find specific invariant solutions. The classification of admitted point symmetries exhibited in this paper would allow one to construct such exact solutions, including new solutions not obtainable from considering only admitted point symmetries of the nonlinear wave equation. In particular, important exact solutions obtained for the linear wave equation for bounded wave speeds [5] ($c' = c^2\nu^{-1}\sin(\nu \log c)$ in **XT**) induce (through the appropriate nonlocal transformation) nontrivial exact solutions of the nonlinear wave equation.

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Appendix A. Determining equations for symmetries M_7, M_8 (Table 4)

The symmetry generators M_7, M_8 have the form

$$M_{7,8} = \phi\frac{\partial}{\partial q} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u} + \psi\frac{\partial}{\partial v},$$

where $\phi = \alpha q + \beta t$, $\tau = \gamma q + \delta t$, and $\alpha, \beta, \gamma, \delta, \eta, \psi$ are functions of (u, v). The determining equations yielding M_7, M_8 are given by

$$\gamma = \frac{\beta}{v^2 c(u)^2},$$

$$\eta = 3u\frac{v(\delta - \alpha) + \psi}{2v},$$

$$\alpha_u = \frac{v^4(h + v^2)(\alpha - \delta) - 3uh(h - 2v^2)\beta + v^3(h + v^2)\psi}{3uv^2h^2},$$

$$\alpha_v = \frac{-v^2(h^2 + 2v^2h + v^4)(\alpha - \delta) + 3uh(h - 2v^2)\beta - v(h^2 + 2v^2h + v^4)\psi}{v(h + v^2)h^2},$$

$$\beta_u = \frac{-v^2(h^2+2v^2h+v^4)(\alpha-\delta) - 3uh(3h+2v^2)\beta - v(h^2+2v^2h+v^4)\psi}{9u^2h^2},$$

$$\beta_v = \frac{v^4(h+v^2)(\alpha-\delta) + 6uh(v^2+h)\beta + v^3(h+v^2)\psi}{3uvh^2},$$

$$\delta_u = \frac{v^2(3v^2h+2v^4+h^2)(\alpha-\delta) + 6uh(h+2v^2)\beta + v(3v^2h+2v^4+h^2)\psi}{6uv^2h^2},$$

$$\delta_v = \frac{-v^2(3v^2h+2v^4+h^2)(\alpha-\delta) - 6uh(2v^2+3h)\beta - v(3v^2h+2v^4+h^2)\psi}{2v(h+v^2)h^2},$$

$$\psi_u = \frac{v^2(h+v^2)(\alpha-\delta) - 12\beta uh + v(h+v^2)\psi}{6uvh},$$

$$\psi_v = \frac{v^3(\alpha-\delta) + (v^2+2h)\psi}{2vh},$$

where $h = h(u, v) = 9u^2c^2(u) - v^2$, and $c(u) = u^{-2/3}$.

Appendix B. Determining equations for symmetries N_5, N_6 (Table 5)

The symmetry generators N_5, N_6 have the form

$$N_{5,6} = \rho\frac{\partial}{\partial r} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u} + \psi\frac{\partial}{\partial v},$$

where $\rho = \alpha r + \beta t$, $\tau = \gamma r + \delta t$, and $\alpha, \beta, \gamma, \delta, \eta, \psi$ are functions of (u, v). The determining equations yielding N_5, N_6 are given by

$$\gamma = \frac{\beta}{v^2u^2c(u)^2},$$

$$\eta = \frac{3vu\delta - 3\alpha vu + 3\psi u)}{v},$$

$$\alpha_u = \frac{2v^4(h+v^2)(\alpha-\delta) - 3h(h-2v^2)\beta - 4v^3(h+v^2)\psi}{3uv^2h^2},$$

$$\alpha_v = \frac{2v^2(2h^2+3v^2h+v^4)(\alpha-\delta) - 3h(h-2v^2)\beta - 4v(2v^2h+v^4+h^2)\psi}{v(h+v^2)h^2},$$

$$\beta_u = \frac{2v^2(2h^2+3v^2h+v^4)(\alpha-\delta) + 6\beta v^2h - 4v(2v^2h+v^4+h^2)\psi}{9uh^2},$$

$$\beta_v = \frac{2v^4(h+v^2)(\alpha-\delta) + (6v^2h+6h^2)\beta - 4v^3(h+v^2)\psi}{3vh^2},$$

$$\delta_u = \frac{-2v^2(h^2-v^4)(\alpha-\delta) + 3h(h+2v^2)\beta + 4v(h^2-v^4)\psi}{3uv^2h^2},$$

$$\delta_v = \frac{2v^2(2h^2+3v^2h+v^4)(\alpha-\delta) - 3h(3h+2v^2)\beta - 4v(3v^2h+v^4+2h^2)\psi}{v(h+v^2)h^2},$$

$$\psi_v = \frac{-v^3\alpha + v^3\delta + (h+2v^2)\psi}{vh},$$

$$\psi_u = \frac{(v^2h+v^4)\alpha - 6\beta h + (-v^2h-v^4)\delta + (-2vh-2v^3)\psi}{3uvh},$$

where $h = h(u, v) = 9u^2c^2(u) - v^2$, and $c(u) = u^{-4/3}$.

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