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BECKMANN'S EDGEWORTH-BERTRAND DUOPOLY EXAMPLE REVISITED

ALEXEI F. CHEVIAKOV

Department of Mathematics, University of British Columbia, Vancouver, B.C. V6T 1Z4

JOHN M. HARTWICK

Department of Economics, Queen's University Kingston, Ontario K7L 3N6

In the Edgeworth-Bertrand price game, each player has a capacity output, faces the same market demand, and calls out a price. The high-price caller gets some residual market at her price. The low-price caller gets her capacity at her price or all of the market. We re-work Beckmann's closed form solution to his symmetric version of this game, mostly in mixed strategies, and observe that expected price played by a player declines with the size of her exogenously given capacity.

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1. Introduction

The Bertrand-Edgeworth duopoly price game has two players with fixed upper bounds on costless production who each simultaneously call out a price, given a market demand schedule. The low price caller gets either her capacity production as her equilibrium output or all of market demand at that price. The high price caller gets a specific residual quantity as her output at her high price, given positive excess demand at the low price, or she gets zero. In a Nash equilibrium with mixed strategies, expected profit to player i is the same and maximal for any price call in her solution set of prices, "against" the mixed strategy price calls of player j, and similarly for player j. Beckmann (1967), with Dieter Hochstadter, set out a closed form solution for such a game, given a linear (slope −1) market demand schedule (price intercept at 1) and each player with the same bound on her production, i.e., they solved for equilibrium distribution functions on price "calls" for each possible admissable capacity possessed by the two players. They determined that for small capacities, each player would call out the joint monopoly price as a pure strategy.

For large capacities, each would mix her price calls according to an equilibrium distribution function. A close reading of the paper reveals it to be curiously incomplete and in fact flawed in an interesting way.[1] **We observe that the average price called out declines with the sum of the capacities of the players, where this average price includes the pure strategy calls.**[2] Beckmann in contrast inferred a jump up in average price, at the capacity margin separating the pure strategy range of price calls from the mixed strategy range. Specifically, the characterization, sketched in Beckmann's Fig. 3, which we establish is correct, does not follow from the formulas he derived. Here we repair the Beckmann analysis and fill in the missing pieces and hopefully provide the last word on Beckmann's interesting example. We end up with (a) pure strategy equilibria (joint monopoly) for small capacities (b) mixed strategy equilibria with market sharing for medium capacities (capacities sum to less than $\frac{1}{2}$, and (c) mixed strategy equilibria with possible market "pre-emption" for large capacities. In this last case, the high price player ends up on occasion with no market or has been "pre-empted" by the low price player. In the limit of each player with capacity very near 1, each player plays her "pre-emption" price, almost surely.

As Beckmann noted and we observe, for small capacities, the players do not randomize and end up splitting the market between them, each at her capacity, in a joint-monopoly solution. Then for the largest capacity, they do almost no randomization (play their "capacity price" almost surely), and end up in a contestabiltiy equilibrium, one player supplying the complete market at a zero price, at each play. (It is in fact a one shot, simultaneous move game.) When the capacities are such that the joint monopoly price is below $\frac{1}{2}$ (the market demand has intercept, 1 and slope -1) but near $\frac{1}{2}$, the players mix their price calls but the range of price calls is between $\frac{1}{2}$ and the lower support, marginally below $\frac{1}{2}$. It is near the joint-monopoly solution. Larger capacities lead to increased competitive price under-cutting and the average price called by a player moves down farther from $\frac{1}{2}$ as capacity increases. Price $\frac{1}{2}$ remains the joint-monoply price for such cases. Hence larger capacities drive the average price played by a player farther from the joint-monopoly price. In the limit, the average price played is almost zero, played almost certainly and this is the competitive solution or what we call the contestable market solution.

The joint-monopoly price works as a benchmark in the analysis of each possible value for capacity, c. "Large" capacity leads to an average price played by

[1]At the top of page 64, one has "Now p_0 is determined by condition $H(p_0) = 1$. Figure 3 shows p_0 as a function of c." Figure 3 follows directly in the paper. The p_0 sketched in Fig. 3 does not in fact derive from the analysis immediately preceeding the figure. In fact there is no valid derivation of the p_0 schedule in Fig. 3 in the paper, though the p_0 shedule in Fig. 3 is correct for the model. We infer that this correct schedule was arrived at by an interpolation through three correct points. One key point is the subject of Fig. 2 in Beckmann's paper and the other two points, end points of the schedule, were obtained by straightforward ecnomics reasoning.

[2]Maskin (1986) and Allen and Hellwig (1986) have investigated the nature of general equilibrium under Bertrand price competition. The motivation was to have price adjustment in general equilibrium without an auctioneer.

each player, distant from this benchmark price. "Larger" capacity induces more competitive price under-cutting by each player, under-cutting relative to the joint-monopoly price of $\frac{1}{2}$. As capacity is changed in steps from 0 to unity, the average price called out falls below the joint monopoly price by zero (for $0 > c \geqq \frac{1}{4}$) down to $\frac{1}{2}$. Central to the behavior of price calls relative to the exogenously given capacities is what we call "the sharing rule", namely what quantity the *ex post* high price player gets after the simultaneous price calls are made. Beckmann drew on a rule set out by Shubik in the 1950's (see Leviatan and Shubik (1972)). Roughly speaking this rule implies that the high price caller gets low profits but not necessarily zero profits, consistently. Leviatan and Shubik (1972) introduced a different "sharing rule" and the resulting game solved much more easily and as well, with a quite different "evolution" of average prices with capacity change. We suggest that the original Shubik sharing rule which is at the center of Beckmann's analysis is somewhat more realistic and makes the Beckmann problem well worth revisiting.

2. The Model

Each player has capacity, c, with no costs of production. The market demand schedule is $Q = 1 - P$, where P is "market" price. For p the price of the high price seller and q the price of the low price seller, we have demand, $1 - q$ at the low price. If $1-q$ exceeds player-q's c, then fraction $\frac{1-q-c}{1-q}$ of player-p's potential quantity, $1-p$ is supplied by player-p at price p. That is,

SHARING RULE[3]: In the case of excess demand at lower price q, we observe player-q's quantity supplied to be c, and player-p's quantity supplied to be $\left[\frac{1-q-c}{1-q}\right]$ $[1-p]$, at price p.

If $c \geqq (1-q)$, then player-p supplies nothing.[4]

It is easy to see that each player plays her price $1-2c$ with certainty for the cases $0.25 \geqq c > 0$, and supplies her quantity c. To check this, we see that undercutting by one player with a price deviation (playing a slightly lower price, *ab initio*) yields no increase in output and thus results in a lower revenue (profit). Consider now a deviation upward to price, $[1-2c+\varepsilon]$ by player-p from current price $1-2c$ for ε small

[3]This rule for allocating demand to the high price player is due to Shubik. See Leviatan and Shubik (1972; p. 118). The Leviatan-Shubik (1972) analysis is based on the assumption that the high price player's quantity is the total residual demand at that high price. Cournot price calls become a benchmark for their analysis. And Cournot price calls are the perfect equilibrium in the Kreps-Scheinkmann (1983) two stage game.

[4]We can say that player-p is pre-empted, and left with zero market and zero profit. This pre-emption does not occur for cases of small capacity (c up to $\frac{1}{2}$). And such pre-emption was not dealt with correctly by Beckmann in his derivations for cases of large capacity. Beckmann ended up then with two errors: for small capacities, he used the wrong upper support of his distribution function and for large capacities, he mis-handled the pre-emption issue. Somewhat by chance, for $c = \frac{1}{2}$, he proceeded correctly and he ended up with three correct points, from which he interpolated to his correct Fig. 3, his (p_0, c) schedule. (We infer that he arrived at his two end points from purely economics arguments.)

and positive. Her revenue changes from $\left[\frac{c}{2c}\right](2c)[1-2c]$ to $\left[\frac{c}{2c}\right][2c-\varepsilon][1-2c+\varepsilon]$, given the sharing rule above. The new revenue minus the old is $-\varepsilon+4c\varepsilon-\varepsilon^2$ which is negative for $0.25 \geqq c \geqq 0$. Thus each player playing price equal to $1-2c$ is a Nash equilibrium strategy for the cases $0.25 \geqq c \geqq 0$.

We now pursue this idea of an equilibrium in pure strategies for a value of c slightly in excess of $\frac{1}{4}$. Consider each player playing price equal to $1-2(0.25+\varepsilon)$ for current $c = 0.25+\varepsilon$ for ε small and positive. Profit for each player is $\left[\frac{c}{2c}\right][2c][1-2c]$. Now consider the case of player-p playing a slightly higher price, namely $1-2[0.25+\varepsilon]+\varepsilon$. This price will be below 0.5. Player-p's profit will now be $\left[\frac{c}{2c}\right][2[0.25+\varepsilon]-\varepsilon][1-2[0.25+\varepsilon]+\varepsilon]$. If one subtracts player-p's "original" profit (given player-p's price at $1-2c$) from this profit, one obtains $\frac{3\varepsilon^2}{2}$ which is of course positive. Hence each player playing price $1-2c$, $c = 0.25+\varepsilon$, is not a Nash equilibrium in pure strategies and this suggests searching for a Nash equilibrium in mixed strategies for the case of $c > 0.25$. However, before taking up mixed strategies, we address Beckmann's limit-value error.

He inferred that for the case of say, $c = 0.25+\varepsilon$, the range of prices played in the mixed strategy solution should terminate at upper bound, $1-c$. We reject this and make the case for an upper limit of $\frac{1}{2}$, which we defend below with a formal argument turning on the **maximization** of expected profit. But it is obviously useful to obtain some intuition on this crucial point. To this end, let us return to our example immediately above and consider player-p "deviating" from the original position (each playing price $1-2c$) with a new price above $\frac{1}{2}$. That is, instead of having her price call at ε above the original price, let it be 3ε above the original. When we carry out the calculation of profit to player-p in the new position, we get $\frac{1}{2}\{-2[0.25+\varepsilon]+8\varepsilon[0.25+\varepsilon]-3\varepsilon-9\varepsilon^2\}$. When her profit in the original position is subtracted from this, we get $-\varepsilon-\varepsilon^2$, which is negative. This suggests, but does not prove, that in a mixed strategy equilibrium for the case $c = 0.25+\varepsilon$, the range of prices played will be bounded above by $\frac{1}{2}$.

And a last intuition on this point of difference with Beckmann. We agree with Beckmann that for $c = 0.25$, it is correct to infer that each player plays price $1-2c$ with certainty. (Below we obtain this result when we take $c = 0.25+\varepsilon$ and move ε down to zero.) Hence the spike of probability mass at price $1-2c$ for $c = 0.25$. Beckmann's assumption of the upper limit of the density at $1-c$ for $c = 0.25+\varepsilon$ implies a jump in the upper support of the density for a small increase in c. Such an outcome seems counter-intuitive to us and in fact can be "ruled out" under the assumption that the upper support remains at $\frac{1}{2}$ for both $c = 0.25$ and $c = 0.25+\varepsilon$, for ε small and positive. The smooth shrinking of the density function, as $c \to \frac{1}{4}$ from above, to a spike at price $\frac{1}{2}$ is a central outcome of our investigation of Beckmann's specific formulation of the Edgeworth-Bertrand price game.

We turn now to consider mixed strategy equilibria.

The expected payoff of a pure strategy p by player-p against a mixed strategy $G(q)$ of player-q is

$$v(p,G) = v_1(p) + v_2(p) + v_3(p),$$

where

$$v_1(p) = p\int_{p_0}^{\min(p,1-c)} \min\left(c, \left[\frac{1-q-c}{1-q}\right][1-p]\right) dG(q), \quad \{p > q \text{ and } p_0 < 1-c\}$$
$$v_2(p) = p\min\left(c, \frac{1-p}{2}\right) dG(p), \quad \{p = q\}$$
$$v_3(p) = p\min(c, 1-p)\int_p^{p_1} dG(q), \quad \{p < q\} \tag{1}$$

where $dG(q) = g(q)dq$, and $g(q)$ is the density function, the same for each player. $g(p)$ is zero for p outside the range (p_0, p_1) and $1 \geqq p_1, p_0 \geqq 0$. The first term of (1) corresponds to the case $p > q$ and is zero if $p_0 \geqq 1-c$; the second term $v_2(p)$ corresponds to case $p = q$; the third term $v_3(p)$ corresponds to $p < q$.

Continuous equilibrium has (1) each player playing from the same set of strategies (distribution functions, to be solved for, are the same), (2) $\forall\, p$, $v(p, G) = \bar{v} =$ constant, (3) $\bar{v}$ is maximal over admissable $v(p, G)$ and (4) $G(p)$ is a continuous function.[5]

Before turning to solving for $G(p)$, we set out three useful preliminary results.

Lemma 1. *For* $\frac{1}{2} \geqq c \geqq \frac{1}{4}$, *and* $G(p)$ *continuous on* (p_0, p_1), *the strategy,* $p = 1-c$ *dominates* $p > 1-c$.

Proof. Suppose that $p \geqq 1-c$ and consider the three terms in (1) separately.

(1) For $p > q$, we have player-p's expected profit $v_1(p) = p\int_0^{\min(p,1-c)}\left(c, \frac{1-c-q}{1-q}(1-p)\right)dG(q)$. In this case we have $\frac{1-q-c}{1-q}(1-p) = \left(1-\frac{c}{1-q}\right)(1-p) \leqq \left(1-\frac{c}{1-q}\right)c \leqq c$ since $0 < \frac{c}{1-q} < 1$. Hence $p\int_{p_0}^{1-c}(1-p)\frac{1-q-c}{1-q}dG(q) = p(1-p) \times \mathit{const}$. Hence this latter increases as p decreases to $1-c$ from above.

(2) For $p = q$ we have expected profit $dG(p)\min\left(c, \frac{1=p}{2}\right)p = 0$ since $dG(p) = 0$ for all p.

(3) Look at $p < q$. We have $v_3(p) = p\min(c, 1-p)\int_p^{p_1} dG(q) = p(1-p)\int_p^{p_1} dG(q)$. Since $1-c \geqq \frac{1}{2}$ for $\frac{1}{2} \geqq c \geqq \frac{1}{4}$, both $\int_p^{p_1} dG(q)$ and $p(1-p)$ increase when p decreases down to $1-c$, therefore for $p \geqq 1-c$ $v_3(p)$ is maximized at $p = 1-c$

□

Lemma 2. *For* $1 \geqq c > \frac{1}{2}$, *and* $G(p)$ *continuous on* (p_0, p_1), *the strategy* $p = \frac{1}{2}$ *dominates those with* $p > \frac{1}{2}$.

[5] For Beckmann's problem, obtaining a maximum for expected profit can be reduced to, getting the limits of the equilibrium distribution function correct. Beckmann intuited the upper support of the distribution function for small capacities and got it wrong. He neglected to consider maxima of expected profit. Invoking maximization removes the need to intuit.

Proof. We again consider the three terms in (1).

(1) For $p > q$ and $p > \frac{1}{2}$, player-p's expected profit becomes $v_1(p) = \int_{p_0}^{1-c} \frac{1-c-q}{1-q} dG(q) \times p(1-p) = p(1-p) \times const$. This latter increases as p approaches $\frac{1}{2}$ from above.
(2) For $p = q$, profit is zero because $G(p)$ is continuous.
(3) For $p < q$, player-p's expected profit becomes $v_1(p) = A_1 B_1$ for $A_1 = p(1-p)$ and $B_1 = \int_p^{p_1} dG(q)$. We observe that both A_1 and B_1 increase as p approaches $\frac{1}{2}$ from above. □

Lemma 3. $\bar{v} = p_0 c$, *given* $G(q)$ *continuous*, $p \in (p_0, p_1)$; $1 \geqq p_0, p_1 \geqq 0$, $\forall c \in \left(\frac{1}{4}, 1\right)$.

Proof. (a) $\frac{1}{2} \geqq c > \frac{1}{4}$. From Lemma 1, p_0, $p_1 \leqq 1 - c$, therefore $c \geqq 1 - p_0$. By the above definition of equilibrium, $v(p, G) = \bar{v}$ for any p. This implies that $\bar{v} = v(p_0, G) = cp_0 + 0$.

(b) $1 > c > \frac{1}{2}$. By Lemma 2, $p_0 \leqq \frac{1}{2}$. Consider 2 cases: $p_0 > 1 - c$ and $p_0 \leqq 1 - c$.

(i) If $p_0 > 1 - c$, then we find $\bar{v}_1 = v(p_0, G) = p_0 \cdot (1 - p_0)$.
(ii) If $p_0 \leqq 1 - c$, then $\bar{v}_2 = v(p_0, G) = cp_0$.

We have to determine which of (i), (ii) we should use at each value of c. Suppose that for some $c = c^*$ case (i) is valid, i.e., $p_0 > 1 - c^*$ and $\bar{v} = p_0 \cdot (1 - p_0)$. Then in formula for profit 1 $v_1(p) \equiv 0$. Then we can solve this case completely, getting $1 - G(p) = \frac{p_0 \cdot (1-p_0)}{p \cdot (1-p)}$. This formula suggests that $G(p)$ is not a probability distribution function suitable for this problem, because $\forall p \in [0, 1]$ $G(p) < 1$. ($G(p) \to 1$ only at $p \to \infty$). This implies that only case (ii) is valid, i.e., for all $c \in (\frac{1}{4}, 1)$ $p_0 \leqq 1 - c$ and $\bar{v} = cp_0$. □

3. Characterization of G(p)

Our analysis for small $c's$ is then similar to Beckmann's, except we "digress" , with our argument concerning the maximization of expected profit, to establish that the upper support of $G(p)$ is $\frac{1}{2}$. We make use of our three lemmas.

(1) For $\frac{1}{2} \geqq c > \frac{1}{4}$ we have expected profit

$$cp_0 = pc \int_p^{p_1} dG(q) + p(1-p) \int_{p_0}^{p} \frac{1-c-q}{1-q} dG(q). \tag{2}$$

We denote $H(p) = 1 - G(p)$. We differentiate by p, in (2), and obtain[6]

$$H'(p) + \frac{cH(p)}{(1-p)(2c+p-1)} + \frac{cp_0(1-2p)}{p^2(1-p)(2c+p-1)} = 0. \tag{3}$$

[6]Getting ahead of ourselves, one observes here that if, for $p \to \frac{1}{2}$ from below, $H'(p)$ and $H(p)$ each approach zero, this key equation is satisfied. This value of p, turns out in our analysis below to be the upper support of the distributions which define each player's equilibrium strategy.

(2) For $1 > c > \frac{1}{2}$, $v(p, G)$ is generally in two parts, corresponding to (a) $p \geqq 1-c$ and (b) $p < 1 - c$.[7] For case (b) we have

$$cp_0 = p(1-p)\int_{p_0}^{p} \frac{1-q-c}{1-q} dG(q) + cp\int_{p}^{p_1} dG(q). \quad 1-c \geqq p > p_0$$

This coincides with expected profit in (2) and hence for this case Eq. (3) is valid.

Case (a) involves player-q sometimes leaving player-p with zero market and profit. Expected profit for this case (a) is

$$cp_0 = p(1-p)\int_{p_0}^{1-c} \frac{1-q-c}{1-q} dG(q) + (1-p)p\int_{p}^{p_1} dG(q), \quad p_1 > p > 1-c$$

if $p_1 > 1 - c$. Therefore we have

$$G(p) = 1 + A - \frac{cp_0}{(1-p)p} \tag{4}$$

for

$$\begin{aligned} A &= \int_{p_0}^{1-c} \frac{1-q-c}{1-q} dG(q) \\ &= const. \end{aligned} \tag{5}$$

The $G(p)$ for $p \in (p_0, 1 - c)$ relevant for expression A is derived below.

3.1. *Profit maximizing solution for* $c = \frac{1}{2}$

Since $G(p)$ is a probability distribution function, it must be continuous and non-decreasing on (p_0, p_1). We solve (3) for $c = \frac{1}{2}$ and $H(p_1) = 0$ (i.e., $G(p_1) = 1$). One gets

$$G(p) = 1 - \frac{p_0}{3}\left[\left(4 - \frac{2}{p} + \frac{1}{p^2}\right) - \left(4 - \frac{2}{p_1} + \frac{4}{p_1^2}\right)\left(\frac{p_1}{1-p_1}\frac{1-p}{p}\right)^{1/2}\right]. \tag{6}$$

Putting $G(p_0) = 0$, yields

$$1 = \frac{p_0}{3}\left[\left(4 - \frac{2}{p_0} + \frac{1}{p_0^2}\right) - \left(4 - \frac{2}{p_1} + \frac{4}{p_1^2}\right)\left(\frac{p_1}{1-p_1}\frac{1-p_0}{p_0}\right)^{1/2}\right]. \tag{7}$$

This defines curve $p_0(p_1)$ which investigation reveals to be increasing. It follows that $G(p)$ is also increasing in $(0, 1)$, given a pair (p_0, p_1). Expected profit cp_0 is a maximum for p_0 a maximum and this occurs at $p_1 = \frac{1}{2}$. Corresponding to this value of p_1 one obtains $p_0 = 0.172$.[8]

[7] We established in Lemma 3 that $p_0 < 1 - c$.

[8] This was obtained by Beckmann because his upper support, $1 - c$ occurs at $\frac{1}{2}$, which we have verified is the correct numerical value, though $1-c$ is not the correct support for other values of c.

3.2. *Profit maximizing solution for* $\frac{1}{2} > c > \frac{1}{4}$

We solve Eq. (3) with $G(p_1) = 1$ and impose the condition $G(p_0) = 0$. The solution for $H(p) = 1 - G(p)$ is

$$\begin{aligned} H(p) &= \Phi(p)\frac{cp_0(1-p)^{1/2}}{(2c-1+p)^{1/2}}, \\ \Phi(p) &= \int_p^{p_1} \frac{1-2q}{q^2(1-q)^{3/2}(2c+q-1)^{1/2}}dq. \end{aligned} \tag{8}$$

We forego writing out the integral here. That $G(p_1) = 1$, is used in obtaining (8). The requirement that $G(p_0) = 1$ yields the dependence in $p_0(p_1)$. We illustrate this in Fig. 1 for the case $c = 0.35$. It is straightforward to see that such a curve has one maximum for each $c \in \left(\frac{1}{4}, \frac{1}{2}\right)$. And this maximum always occurs at $p_1 = \frac{1}{2}$. To see this, we substitute $p_0 = p_0(p_1)$ into condition $1 - H(p_0) = 0$, and differentiating by p_1, and setting the result to zero, we have $cp_0(1-2p_1)(1-p_0(p_1))^{1/2} = 0$. The only satisfactory solution is $p_1 = \frac{1}{2}$ and this result is independent of the value of c. Then given $p_1 = \frac{1}{2}$, we obtain p_0 in $1 - H(p_0) = 0$. The corresponding solution is profit maximizing for our chosen values of c, since p_1 was chosen to make p_0 a maximum. We proceeded to calculate p_0 for various values of c (as in Beckmann's Fig. 3), in Fig. 2. We have also inserted the mean price $\bar{p}$ in Fig. 2. Each density

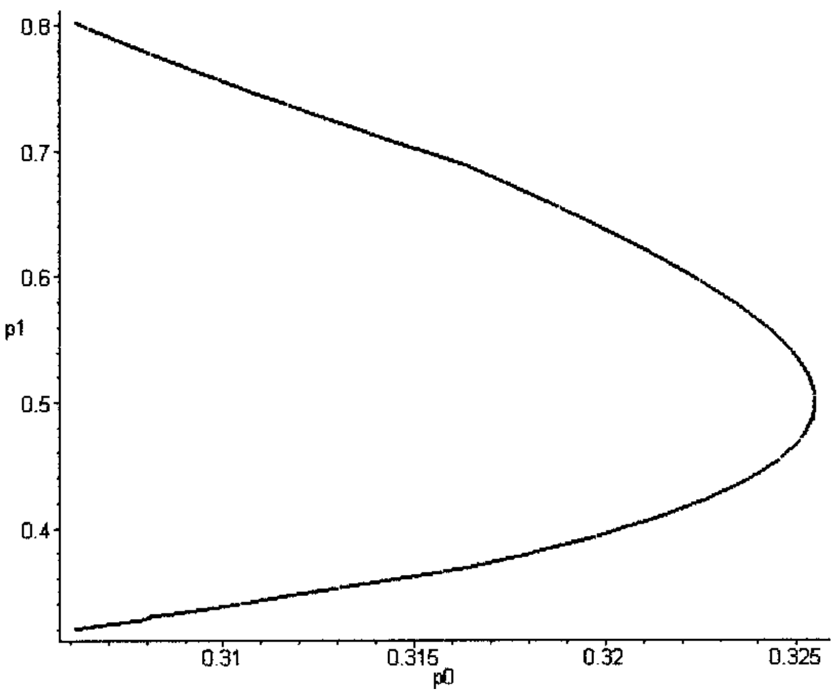

Fig. 1.

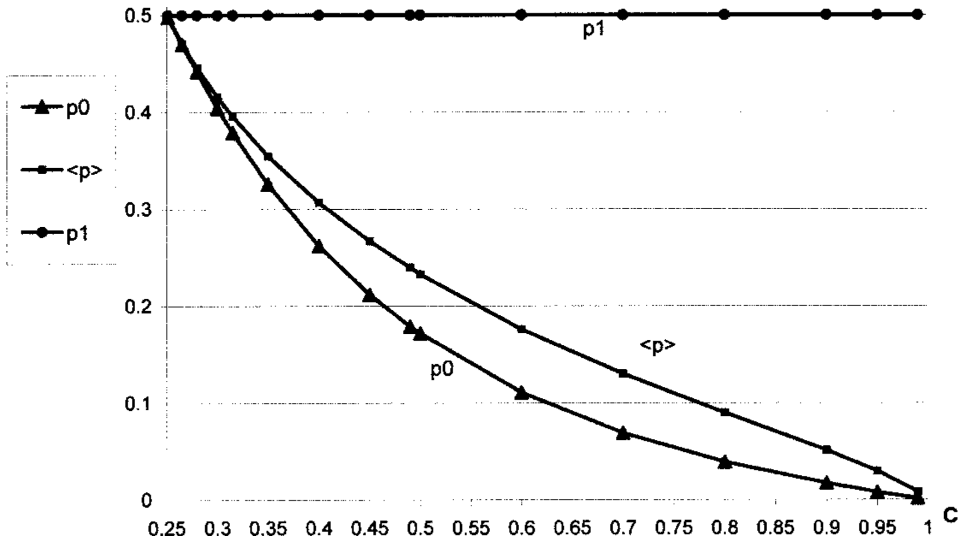

Fig. 2.

function peaks at the lower support and declines smoothly to zero at upper support, $\frac{1}{2}$. The mean price call is then generally relatively close to the lower support. The variance peaks for $c \cong 0.7$. The ability of each player to under-cut with her price relative to the joint monopoly price of $\frac{1}{2}$ leads to average price calls relatively distant from price $\frac{1}{2}$ as c is set farther above $\frac{1}{4}$. Price competition is bad for potential joint monopoly profits. The ability of each player to commit to price $\frac{1}{2}$ would give them a joint profit maximum.

3.3. *Profit maximizing solution for* $1 > c > \frac{1}{2}$

This case involves possible pre-emption as when the lower price caller wins all of the prospective market, leaving the other player with no market and no profit. This pre-emption occurs because, roughly speaking, price, $\frac{1}{2}$ yields joint monopoly profit and there are no costs of playing for a part of the market less than one's current capacity, c. Each player now has an incentive to play prices up to $\frac{1}{2}$ even though she knows she will be, in ideal conditions, getting a part of the market less than her current c. Above, with small capacities, no player ever played a price above $1 - c$. Here with large capacities, it is rational for each player to occasionally call out a price at $1 - c$ or above (up to $\frac{1}{2}$). Beckmann was aware of this difference between small and large capacity cases, but did not work out the large capacity case correctly. For this case of large capacities, the complete $G(p)$ function for each c involves two pieces, appropriately joined at price $1 - c$. Below we label these two pieces, $G_1(p)$ and $G_2(p)$. We turn to some formal details for player-p's call below and above price, $1 - c$.

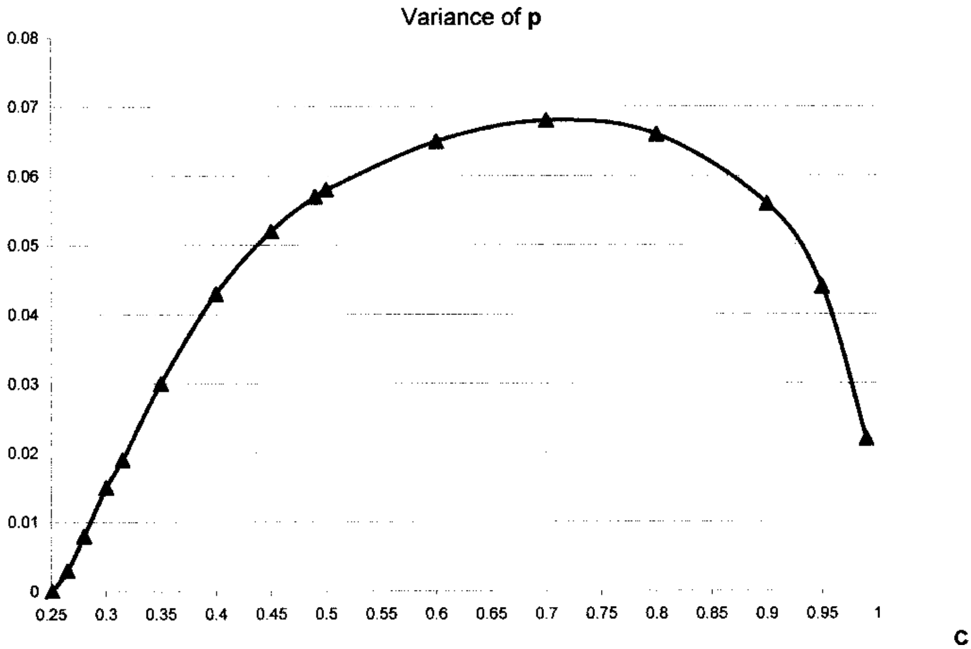

Fig. 3.

Consider two cases.

(a) $\min(p_1, 1-c) \geqq p > p_0$. When player-$p$ makes a price call below $1-c$, her expected profit is "the familiar"

$$cp_0 = p(1-p)\int_{p_0}^{p} \frac{1-q-c}{1-q} dG(q) + cp\int_{p}^{1/2} dG(q), \tag{9}$$

and the equation for $G(p)$ coincides with (3) above.

(b) $p_1 > p \geqq 1-c$, if $p_1 > 1-c$. Here, player-p's current price call can be at or above $1-c$, and her expected profit is[9]

$$cp_0 = p(1-p)\int_{p_0}^{1-c} \frac{1-q-c}{1-q} dG(q) + (1-p)p\int_{p}^{1/2} dG(q), \quad \frac{1}{2} \geqq p \geqq 1-c. \tag{10}$$

Now $G(p) = 1 + A - \frac{cp_0}{p(1-p)}$, where A is defined following (4, 5). Note that from the proof of Lemma 3 we know $p_0 < 1-c$.

We now establish that p_1, the upper support, is never less than $1-c$. This means that both cases (a), (b) take place at any $c \in \left(\frac{1}{2}, 1\right)$.

[9]We have seen this and the next equation above with $\frac{1}{2}$ here replaced by p_1 above in the limits. So our primary task here is to establish proper values for p_1 and p_0. In particular that p_1 is indeed $\frac{1}{2}$.

First look at $p_1 < 1 - c$. Then only case (a) above is relevant, and p is in the range $p_0 < p < p_1 < 1 - c$. Thus the complete solution for $p \in (p_0, p_1)$ is defined by (3) and is analogous to the case with $\frac{1}{2} > c > \frac{1}{4}$. Using the condition $G(p_0) = 0$, we obtain implicit dependence $p_0 = p_0(p_1)$. It can be analytically shown, through involved computations, that on $p_1 \in (0, 1-c)$ $p_0(p_1)$ is a strictly increasing function, therefore maximum of $p_0(p_1)$, which is maximum of profit, is not reached on the interval $p_1 \in (0, 1-c)$. But if we continue the curve $p_0(p_1)$ allowing $p_1 > 1 - c$ and making $G(p)$ a piecewise function

$$G(p) = \left\{ \begin{array}{l} G_1(p),\ p_1 \geqq p \geqq 1 - c \\ G_2(p),\ 1 - c \geqq p \geqq p_0, \end{array} \right\}, \tag{11}$$

with the conditions $G_1(1 - c) = G_2(1 - c)$, $G_2(p_0) = 0$, $G_1(p_1) = 1$, then it is possible to show, in the way similar to case $\frac{1}{2} > c > \frac{1}{4}$, that maximal profit is reached at $p_1 = \frac{1}{2} > 1 - c$.

We summarize. For a "large" c, we always have $p_0 < 1 - c$, $p_1 = \frac{1}{2} > 1 - c$, and $G(p)$ is a continuous composition (11) of two density functions that are solutions to different differential equations (9, 10). We also plotted the means of each density function in Fig. 2. The mean values are relatively close to the values of p_0 since each density function is convex and peaks at its lower support. Each is asymptotic to zero at the upper support of $\frac{1}{2}$. A central result of our analysis is that expected price called out by a player is always declining in her capacity. Relatively abundant quantity, *ex ante*, results in relatively low prices, *ex post*.

4. Concluding Remark

As we noted above, a characterization of the solution is that a larger capacity leads to a lower price, either certain or expected. Each player would like to end up at the joint monopoly position but for $c > \frac{1}{4}$, they cannot attain the joint monopoly position because of the credible threat of price undercutting by the "partner". The larger is c the worse is the effect of competitive price undercutting by each player, relative to the potential joint monopoly profit outcome with each player playing a price of $\frac{1}{2}$. If they could communicate and reach a binding agreement, then each would play price $\frac{1}{2}$ for $c > \frac{1}{4}$. Hence strategic uncertainty is pro-competition in the sense that the average price played by either player is always below $\frac{1}{2}$ for $c > \frac{1}{4}$. The larger is c the farther the credible threat of price undercutting takes the average price played by a player from the monopoly price of $\frac{1}{2}$. With $c = 1$ (the competitive case), only one player produces and the market price is almost surely zero, and so this "competitive outcome" could be characterized as a contestability equilibrium.

We have not proved that for each player to play with the same distribution function is the optimal strategy. It is difficult to see how an "asymmetric" game would work here. We presume then that symmetry is optimal here. Beckmann reflected on a game with the two players with "unequal capacities c_1 and c_2. The integral equation is a straightforward generalization [of the above]. However, its solution may no longer be developed in closed form." (p. 67) We believe that for the case of distinct but very similar capacities that the solution will be very similar to what we have worked out above, for the case of identical capacities. Small capacities would involve pure strategies. There are two opposing forces at work with c_1 not equal to c_2. The player with the larger capacity has more to gain from having the joint monopoly price obtain and hence would appear less motivated to undercut in price. On the other hand, having a relatively large capacity would encourage one not to be price undercut since the high price "residual" quantity will in general be small. The symmetric game may thus be a small step to a complete analysis.

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