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Exact anisotropic MHD equilibria

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Abstract
An infinite-dimensional Lie group of symmetries of the anisotropic Chew–Goldberger–Low (CGL) plasma equilibrium equations is introduced. The symmetries are used to construct families of new anisotropic plasma equilibria. An infinite-dimensional family of transformations between solutions to the isotropic magnetohydrodynamic (MHD) equilibrium equations and solutions to the anisotropic CGL plasma equilibrium equations is presented. The transformations depend on the topology of the original solutions and produce a wide class of anisotropic plasma equilibrium solutions, including 3D solutions with no geometrical symmetries.

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1. Introduction

The most important continuum plasma models are the isotropic magnetohydrodynamics (MHD) equations [1] and the so-called anisotropic CGL (Chew–Goldberger–Low) magnetohydrodynamics equations [2]. Both are derived from Boltzmann and Maxwell equations under different isotropy assumptions.

The isotropic MHD approximation employs a scalar pressure P and is valid when the mean free path of plasma particles is much less than the typical scale of the problem. When the mean free path for particle collisions is large compared to the Larmor radius (i.e. in strongly magnetized or rarified plasmas), the CGL approximation should be used, where the pressure is a tensor depending on two parameters: $p_\parallel$ and $p_\perp$, which are interpreted as effective pressures along the magnetic field ($p_\parallel$) and in the transverse plane ($p_\perp$).

Non-viscous incompressible and compressible MHD equations, as well as incompressible CGL equations, are known to admit a variational formulation [3].

In the present paper we consider incompressible equilibrium plasma flows and static configurations. In section 2, we present an *infinite-dimensional* set of transformations between isotropic (MHD) and anisotropic (CGL) plasma equilibria. These transformations can be applied to any static plasma equilibrium and to a wide class of dynamic equilibria to yield physically meaningful anisotropic equilibrium solutions.

The transformations depend on the topology of the original isotropic plasma equilibrium. It is known that all isotropic non-viscous incompressible MHD equilibria (except Beltrami flows) have a special topology—the plasma domain is filled with the nested two-dimensional *magnetic surfaces*, on which magnetic field lines and plasma streamlines lie [4–7]. The transformations depend on two arbitrary functions constant on magnetic surfaces. For field-aligned equilibria with magnetic field lines closed or going to infinity, the transformations break the geometrical symmetry of the initial solution.

The presented family of transformations has features different from those for Bäcklund transforms for soliton equations. Unlike Bäcklund transforms, the new transformations have explicit algebraic form and depend on all three spatial variables.

We show that the transformations derived in this paper produce the anisotropic plasma equilibria that satisfy the necessary physical conditions and are stable with respect to the fire-hose and mirror instabilities. In section 3, we construct exact anisotropic plasma equilibrium solutions that model anisotropic astrophysical jets.

The new MHD $\rightarrow$ CGL equilibrium transformations can be applied to any known analytical isotropic MHD models, such as Hill-vortex-like solutions [8, 9], to produce corresponding anisotropic plasma equilibria with the same topology of magnetic surfaces.

The ideal (non-viscous) isotropic MHD equilibrium equations possess an infinite-dimensional Abelian group of symmetries G_m [4, 5] that preserve the solution topology but can break the geometrical symmetries. In section 4, we present the generalization of these symmetries for the case of incompressible ideal anisotropic CGL plasmas. The symmetries form an infinite-dimensional Abelian group G with 16 connected components.

2. Transformations between the MHD and CGL equilibria

In this section we present an infinite-dimensional family of transformations that turn isotropic plasma equilibrium solutions into anisotropic (CGL) ones.

Section 2.1 deals with the dynamic equilibrium case ($\mathbf{V}^2 > 0$). Section 2.2 presents the new transformations for static equilibria. In section 2.3, we discuss the necessary physical conditions for a plasma equilibrium solutions and study the stability of anisotropic equilibrium solutions that can be obtained by the application of the new transformations.

2.1. Transformations for dynamic equilibria

The equilibrium states of isotropic moving plasmas are described by the system of MHD equilibrium equations, which under the assumptions of infinite conductivity and negligible viscosity has the form [1]

$$\rho \mathbf{V} \times \operatorname{curl} \mathbf{V} - \frac{1}{\mu} \mathbf{B} \times \operatorname{curl} \mathbf{B} - \operatorname{grad} P - \rho \operatorname{grad} \frac{\mathbf{V}^2}{2} = 0 \tag{2.1}$$

$$\operatorname{div} \rho \mathbf{V} = 0 \qquad \operatorname{curl}(\mathbf{V} \times \mathbf{B}) = 0 \qquad \operatorname{div} \mathbf{B} = 0. \tag{2.2}$$

Here $\mathbf{V}$ is the plasma velocity, $\mathbf{B}$ is the vector of the magnetic field induction, ρ is the plasma density, P is the plasma pressure and μ is the magnetic permeability coefficient.

In the case of incompressible plasma, the equation

$$\operatorname{div} \mathbf{V} = 0 \tag{2.3}$$

is added to the above system; for a compressible case an appropriate equation of state must be chosen. For example, it can be the adiabatic ideal gas equation of state:

$$P = \rho^{\gamma} \exp(S/c_v) \qquad \mathbf{V} \cdot \operatorname{grad} S = 0. \tag{2.4}$$

Here γ is the adiabatic exponent, c_v is the heat capacity at constant volume and S is the entropy density.

In this paper we restrict our consideration to incompressible plasmas.

Incompressibility condition is widely used in the modelling of plasma media. For example, it is a good approximation for subsonic plasma flows with low Mach numbers $M \ll 1$, $M^2 = \mathbf{V}^2/(\gamma P/\rho)$. For incompressible plasma the continuity equation $\operatorname{div} \rho\mathbf{V} = 0$ implies $\mathbf{V} \cdot \operatorname{grad} \rho = 0$, hence density is constant on plasma streamlines.

It is known [4–7] that all compact incompressible MHD equilibrium configurations, except the Beltrami case $\operatorname{curl} \mathbf{B} = \alpha\mathbf{B}$, $\alpha = \text{const}$, are spanned by two-dimensional magnetic surfaces—the vector fields $\mathbf{B}$ and $\mathbf{V}$ are at every point tangent to magnetic surfaces.

When $\mathbf{V} \| \mathbf{B}$, magnetic surfaces may not be uniquely defined for unbounded configurations with magnetic field lines going to infinity, as well as for configurations with closed magnetic field lines.

For *anisotropic* plasmas with the Larmor radius small compared to the characteristic dimensions of the system, the corresponding set of equilibrium equations is [2]

$$\rho\mathbf{V} \times \operatorname{curl} \mathbf{V} - \frac{1}{\mu}\mathbf{B} \times \operatorname{curl} \mathbf{B} = \operatorname{div} \mathbb{P} + \rho \operatorname{grad} \frac{\mathbf{V}^2}{2} \tag{2.5}$$

$$\operatorname{div} \rho\mathbf{V} = 0 \qquad \operatorname{curl}(\mathbf{V} \times \mathbf{B}) = 0 \qquad \operatorname{div} \mathbf{B} = 0 \tag{2.6}$$

where $\mathbb{P}$ is the pressure tensor with two independent parameters $p_{\|}$, $p_{\perp}$:

$$\mathbb{P}_{ij} = p_{\perp}\delta_{ij} + \frac{p_{\|} - p_{\perp}}{B^2} B_i B_j \qquad i, j = 1, 2, 3. \tag{2.7}$$

For this system to be closed, one needs to add to it two equations of state. In this paper we will consider incompressible CGL plasmas: $\operatorname{div} \mathbf{V} = 0$.

Using vector calculus identities, the pressure tensor divergence may be rewritten in the form

$$\operatorname{div} \mathbb{P} = \operatorname{grad} p_{\perp} + \tau \operatorname{curl} \mathbf{B} \times \mathbf{B} + \tau \operatorname{grad} \frac{\mathbf{B}^2}{2} + \mathbf{B}(\mathbf{B} \cdot \operatorname{grad} \tau) \tag{2.8}$$

$$\tau = \frac{p_{\|} - p_{\perp}}{\mathbf{B}^2}. \tag{2.9}$$

Here τ is the anisotropy factor. Therefore, the system (2.5)–(2.6) can be rewritten as

$$\rho\mathbf{V} \times \operatorname{curl} \mathbf{V} - \left(\frac{1}{\mu} - \tau\right) \mathbf{B} \times \operatorname{curl} \mathbf{B} = \operatorname{grad} p_{\perp} + \rho \operatorname{grad} \frac{\mathbf{V}^2}{2} + \tau \operatorname{grad} \frac{\mathbf{B}^2}{2} + \mathbf{B}(\mathbf{B} \cdot \operatorname{grad} \tau) \tag{2.10}$$

$$\operatorname{div} \mathbf{V} = 0 \qquad \operatorname{div} \mathbf{B} = 0 \qquad \operatorname{curl}(\mathbf{V} \times \mathbf{B}) = 0. \tag{2.11}$$

The following theorem shows that there exist infinite-dimensional transformations that map solutions of incompressible MHD equilibrium equations to incompressible anisotropic (CGL) equilibria.

Theorem 1. *Let* $\{\mathbf{V}(\mathbf{r}), \mathbf{B}(\mathbf{r}), P(\mathbf{r}), \rho(\mathbf{r})\}$ *be a solution of the system (2.1)–(2.3) of incompressible MHD equilibrium equations, where the density* $\rho(\mathbf{r})$ *is constant on both magnetic field lines and plasma streamlines (i.e. on magnetic surfaces* $\Psi = \text{const}$*, if they exist).*

Then $\{\mathbf{V}_1(\mathbf{r}), \mathbf{B}_1(\mathbf{r}), p_{\perp 1}(\mathbf{r}), p_{\parallel 1}(\mathbf{r}), \rho_1(\mathbf{r})\}$ *is a solution to incompressible CGL plasma equilibria (2.10), (2.11), where*

$$\begin{aligned}
&\mathbf{B}_1(\mathbf{r}) = f(\mathbf{r})\mathbf{B}(\mathbf{r}) \qquad \mathbf{V}_1(\mathbf{r}) = g(\mathbf{r})\mathbf{V}(\mathbf{r}) \qquad \rho_1 = C_0\rho(\mathbf{r})\mu/g^2(\mathbf{r})\\
&p_{\perp 1}(\mathbf{r}) = C_0\mu P(\mathbf{r}) + C_1 + (C_0 - f^2(\mathbf{r})/\mu)\mathbf{B}^2(\mathbf{r})/2\\
&p_{\parallel 1}(\mathbf{r}) = C_0\mu P(\mathbf{r}) + C_1 - (C_0 - f^2(\mathbf{r})/\mu)\mathbf{B}^2(\mathbf{r})/2
\end{aligned} \tag{2.12}$$

and $f(\mathbf{r}), g(\mathbf{r})$ *are arbitrary functions constant on the magnetic field lines and streamlines.* C_0, C_1 *are arbitrary constants.*

The proof is given in appendix A.

Remark 1. Under the conditions of the theorem, the anisotropy factor

$$\tau_1 \equiv (p_{\parallel 1} - p_{\perp 1})\big/{\mathbf{B}_1}^2 = 1/\mu - C_0/f^2(\mathbf{r}) \tag{2.13}$$

is also constant on the magnetic field lines and streamlines, and the following relations hold:

$$\begin{aligned}
&p_{\perp 1}(\mathbf{r}) = C_0\mu P(\mathbf{r}) + C_1 - \tau_1(\mathbf{r}){\mathbf{B}_1}^2(\mathbf{r})/2\\
&p_{\parallel 1}(\mathbf{r}) = C_0\mu P(\mathbf{r}) + C_1 + \tau_1(\mathbf{r}){\mathbf{B}_1}^2(\mathbf{r})/2.
\end{aligned} \tag{2.14}$$

Remark 2. We note that the transformations (2.12) preserve the topology of plasma configurations. All CGL solutions obtained from non-Beltrami MHD equilibria using theorem 1 have the same magnetic surfaces as the original MHD equilibrium.

2.2. Transformations for static equilibria

Let us rewrite the above theorem for the case of static plasma equilibria. In the case $\mathbf{V} = 0$, the MHD equilibrium equations (2.1)–(2.3) take the form

$$\operatorname{curl}\mathbf{B} \times \mathbf{B} = \mu \operatorname{grad} P \qquad \operatorname{div}\mathbf{B} = 0 \tag{2.15}$$

and the CGL equations can be rewritten as

$$\left(\frac{1}{\mu} - \tau\right)\operatorname{curl}\mathbf{B} \times \mathbf{B} = \operatorname{grad} p_\perp + \tau \operatorname{grad}\frac{\mathbf{B}^2}{2} + \mathbf{B}(\mathbf{B}\cdot\operatorname{grad}\tau) \qquad \operatorname{div}\mathbf{B} = 0. \tag{2.16}$$

From theorem 1, follows

Corollary 1. *Let* $\{\mathbf{B}(\mathbf{r}), P(\mathbf{r})\}$ *be a solution of the static isotropic plasma equilibrium system (2.15). Then* $\mathbf{B}_1(\mathbf{r}), p_\perp(\mathbf{r}), p_\parallel(\mathbf{r})$ *is a solution to the static CGL plasma equilibrium system (2.16), where*

$$\begin{aligned}
&\mathbf{B}_1(\mathbf{r}) = f(\mathbf{r})\mathbf{B}(\mathbf{r})\\
&p_{\perp 1}(\mathbf{r}) = C_0\mu P(\mathbf{r}) + C_1 + (C_0 - f^2(\mathbf{r})/\mu)\mathbf{B}^2(\mathbf{r})/2\\
&p_{\parallel 1}(\mathbf{r}) = C_0\mu P(\mathbf{r}) + C_1 - (C_0 - f^2(\mathbf{r})/\mu)\mathbf{B}^2(\mathbf{r})/2.
\end{aligned} \tag{2.17}$$

Remark 3. The above corollary can be directly used to construct a wide variety of anisotropic plasma equilibrium solutions of different topologies. Indeed, starting with any harmonic function $h(\mathbf{r}) : \triangle h(\mathbf{r}) = 0$ and using a corresponding vacuum magnetic field $\mathbf{B} = \operatorname{grad} h(\mathbf{r})$, one can construct non-degenerate CGL plasma equilibria.

2.3. Physical conditions and stability of new solutions

Let us describe the natural physical conditions for any isotropic and anisotropic MHD equilibrium solutions.

The solutions in a bounded domain $\mathcal{D}$ with the boundary $\partial\mathcal{D}$ should satisfy the following conditions:

$$\begin{aligned}
&0 \leqslant P|_{\mathcal{D}} \leqslant \mathcal{P}_{\max} \qquad \text{(for anisotropic plasmas, } 0 \leqslant p_{\parallel}|_{\mathcal{D}},\ p_{\perp}|_{\mathcal{D}} \leqslant \mathcal{P}_{\max}\text{)}\\
&0 \leqslant \mathbf{B}^2|_{\mathcal{D}} \leqslant \mathcal{B}^2_{\max} \qquad 0 \leqslant \mathbf{V}^2|_{\mathcal{D}} \leqslant \mathcal{V}^2_{\max} \qquad 0 \leqslant \rho|_{\mathcal{D}} \leqslant \rho_{\max}\\
&\mathbf{n}\cdot\mathbf{B}|_{\partial\mathcal{D}} = 0\\
&\mathbf{n}\cdot\mathbf{V}|_{\partial\mathcal{D}} = 0 \qquad \text{or} \qquad \mathbf{V}|_{\partial\mathcal{D}} = 0.
\end{aligned} \tag{2.18}$$

For an unbounded domain $\mathcal{D}$, the natural conditions are

$$\begin{aligned}
&0 \leqslant P|_{\mathcal{D}} \leqslant \mathcal{P}_{\max} \quad \text{(for anisotropic plasmas, } 0 \leqslant p_{\parallel}|_{\mathcal{D}},\ p_{\perp}|_{\mathcal{D}} \leqslant \mathcal{P}_{\max}\text{)}\\
&0 \leqslant \mathbf{B}^2|_{\mathcal{D}} \leqslant \mathcal{B}^2_{\max} \qquad 0 \leqslant \mathbf{V}^2|_{\mathcal{D}} \leqslant \mathcal{V}^2_{\max} \qquad 0 \leqslant \rho|_{\mathcal{D}} \leqslant \rho_{\max}\\
&P \text{ (or } p_{\parallel},\ p_{\perp}),\ \mathbf{B}^2,\ \mathbf{V}^2,\ \rho \rightarrow \text{const} \quad \text{at} \quad |\mathbf{r}| \rightarrow \infty.
\end{aligned} \tag{2.19}$$

For localized solutions in vacuum, the asymptotic constants must be zero, and the magnetic field $\mathbf{B}$ and the velocity $\mathbf{V}$ should decrease at infinity quickly enough to give *finite total energy*

$$\int_{\mathcal{D}} \left(\frac{\rho\mathbf{V}^2}{2} + \frac{\mathbf{B}^2}{2\mu} \right) \mathrm{d}V < \infty. \tag{2.20}$$

For solutions in vacuum with the domain infinite in a given dimension z (e.g., models of astrophysical jets), the above relations should be satisfied in every layer $z_1 < z < z_2$. All magnetic field lines and plasma current lines should be bounded in the cylindrical radial variable r.

If the free functions $f(\mathbf{r})$, $g(\mathbf{r})$ in the transformations (2.12) are separated from zero, then the transformed anisotropic solutions retain the boundedness of the original solution. The functions $f(\mathbf{r})$, $g(\mathbf{r})$ in every particular model must be chosen so that the new anisotropic solution has proper asymptotics at $|\mathbf{r}| \rightarrow \infty$.

Now we address the question of stability of the anisotropic equilibrium solutions (2.12). Though no universal test of overall stability of MHD and CGL equilibria is available, explicit criteria for certain types of instabilities are known. Under the assumption of double-adiabatic behaviour of plasma [2], the criterion for the fire-hose instability is [10]

$$p_{\parallel} - p_{\perp} > \frac{\mathbf{B}^2}{\mu} \tag{2.21}$$

(or, equivalently, $\tau > 1/\mu$), and for the mirror instability

$$p_{\perp}\left(\frac{p_{\perp}}{6p_{\parallel}} - 1 \right) > \frac{\mathbf{B}^2}{2\mu}. \tag{2.22}$$

Now we explicitly check these conditions for the transformed CGL equilibria $\{\mathbf{V}_1(\mathbf{r}), \mathbf{B}_1(\mathbf{r}), p_{\perp 1}(\mathbf{r}), p_{\parallel 1}(\mathbf{r}), \rho_1(\mathbf{r})\}$ (2.12), supposing that the original isotropic MHD equilibrium configuration $\{\mathbf{V}(\mathbf{r}), \mathbf{B}(\mathbf{r}), P(\mathbf{r}), \rho(\mathbf{r})\}$ satisfies physical conditions (2.18) or (2.19).

From (2.12), for the new solutions

$$p_{\parallel 1} - p_{\perp 1} = \left(\frac{1}{\mu} - \frac{C_0}{f^2} \right) \mathbf{B}_1{}^2 = \frac{\mathbf{B}^2 f^2}{\mu} - C_0\mathbf{B}^2.$$

Hence the fire-hose instability is not present when

$$\frac{\mathbf{B}^2 f^2}{\mu} - C_0\mathbf{B}^2 \leqslant \frac{{\mathbf{B}_1}^2}{\mu} = \frac{\mathbf{B}^2 f^2}{\mu}.$$

Thus any choice of $C_0 \geqslant 0$ prevents the new solutions from having the fire-hose instability.

Now we consider the sufficient condition for the mirror instability (2.22). We define $Q = C_0\mu P(\mathbf{r}) + C_1$, and for stability demand

$$p_{\perp 1}\left(\frac{p_{\perp 1}}{6p_{\parallel 1}} - 1\right) \leqslant \frac{{\mathbf{B}_1}^2}{2\mu}$$

which can be rewritten as

$$-\left(5Q + \frac{7}{2}\left(\frac{f^2}{\mu} - C_0\right)\mathbf{B}^2\right)\left(Q - \frac{1}{2}\left(\frac{f^2}{\mu} - C_0\right)\mathbf{B}^2\right) \leqslant \frac{3f^2\mathbf{B}^2}{2\mu}\left(2Q + \mathbf{B}^2\left(\frac{f^2}{\mu} - C_0\right)\right).$$

This is a quadratic inequality with respect to an unknown function $z = f^2(\mathbf{r})$ constant on magnetic field lines and plasma streamlines:

$$\frac{\mathbf{B}^4}{2\mu}z^2 - 4\mathbf{B}^2(2Q + C_0\mathbf{B}^2)z - \frac{1}{2}\mu(10Q - 7C_0\mathbf{B}^2)(2Q + C_0\mathbf{B}^2) \leqslant 0. \tag{2.23}$$

From this inequality we determine the possible range of $f^2(\mathbf{r})$. If we take $C_1 \geqslant 0$ (and thus $Q \geqslant 0$ for $P \geqslant 0$) and assume $\mathbf{B}^2 \geqslant 0$ in the plasma domain, then the discriminant $D = 3\mathbf{B}^4(2Q + C_0\mathbf{B}^2)(14Q + 3C_0\mathbf{B}^2)$ is non-negative, and the roots are

$$z_{1,2} = \frac{4\mu}{\mathbf{B}^2}(2Q + C_0\mathbf{B}^2) \mp \frac{\mu\sqrt{D}}{\mathbf{B}^4}. \tag{2.24}$$

If the original plasma equilibrium is static, then on every magnetic surface S, $P|_S =$ const $\geqslant 0$, hence $Q|_S =$ const $\geqslant 0$, and it is easy to check that $z_1|_S(|\mathbf{B}|)$ is always concave down, while $z_2|_S(|\mathbf{B}|)$ is concave up. Therefore under the physical assumptions of non-negativity and boundedness of P and $\mathbf{B}^2$, on any magnetic surface S, $\max_S z_1 < \min_S z_2$.

If the original equilibrium is not static, then the function Q is not constant on magnetic surfaces, and it should be explicitly checked that on every surface the inequality $\max_S z_1 < \min_S z_2$ holds.

The values of $f^2(\mathbf{r})$ on magnetic surfaces must be selected within the interval $\max_S z_1 \leqslant f^2(\mathbf{r})|_S \leqslant \min_S z_2$, and thus the new CGL solution will not have the mirror instability. This is the only limitation on the choice of the function $f^2(\mathbf{r})$.

Remark 4. For any MHD equilibrium that satisfies natural physical conditions, by using the transformations (2.12) one can construct infinitely many anisotropic CGL equilibria that are free from the fire-hose instability. Every *static* MHD equilibrium can be transformed into an infinite family of anisotropic equilibria free from the mirror instability.

3. An anisotropic model of astrophysical jets

Below we present anisotropic helically symmetric and non-symmetric models of astrophysical jets. They are obtained by application of the transformations (2.12) to certain isotropic MHD equilibria.

We start with helically symmetric [11] magnetic fields

$$\mathbf{B}_h = \frac{\psi_u}{r}\mathbf{e}_r + B_1\mathbf{e}_z + B_2\mathbf{e}_\phi \qquad B_1 = \frac{\alpha\gamma\psi - r\psi_r}{r^2 + \gamma^2} \qquad B_2 = \frac{\alpha r\psi + \gamma\psi_r}{r^2 + \gamma^2}. \tag{3.1}$$

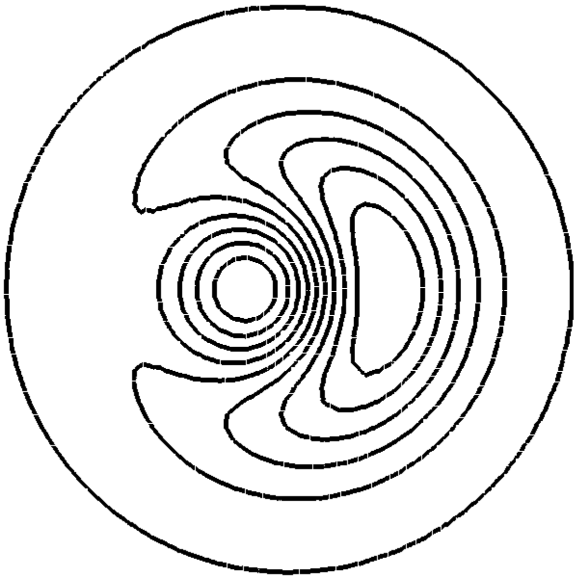


Figure 1. A section of helically symmetric magnetic surfaces. A sample section ($z = 0$) of the magnetic surfaces $\psi(r, \phi) = \text{const}$ for a generally non-symmetric anisotropic astrophysical jet model (section 3). (The parameter values for the plot are $a_1 = -1, \beta = 0.1, \gamma = \sqrt{5/2}$, $\alpha = 3/(2\gamma)$.)

Here $\mathbf{e}_r, \mathbf{e}_z, \mathbf{e}_\phi$ are unit vectors of the cylindrical coordinates (r, z, ϕ), $\psi = \psi(r, u)$ is the flux function, $u = z - \gamma\phi, \alpha = \text{const}, \gamma = \text{const}$. In [12], the exact plasma equilibria (3.1), $\text{curl}\,\mathbf{B} \times \mathbf{B} = \mu\,\text{grad}\,P$, $\text{div}\,\mathbf{B} = 0$ were obtained, which correspond to the flux functions

$$\psi_{Nmn} = \mathrm{e}^{-\beta r^2}(a_N B_{0N}(y) + r^m B_{mn}(y)(a_{mn}\cos(mu/\gamma) + b_{mn}\sin(mu/\gamma))) \tag{3.2}$$

where N, m, n are arbitrary integers $\geqslant 0$ satisfying the inequality $2N > 2n + m$, and $y = 2\beta r^2$. The plasma pressure is $P_h = p_0 - 2\beta^2\psi^2/\mu$, and the plasma velocity $\mathbf{V} = 0$. The functions $B_{mn}(y)$ are polynomials [12].

The simplest exact solution (3.2) is defined for $N = 1, m = 1, n = 0$ and has the form

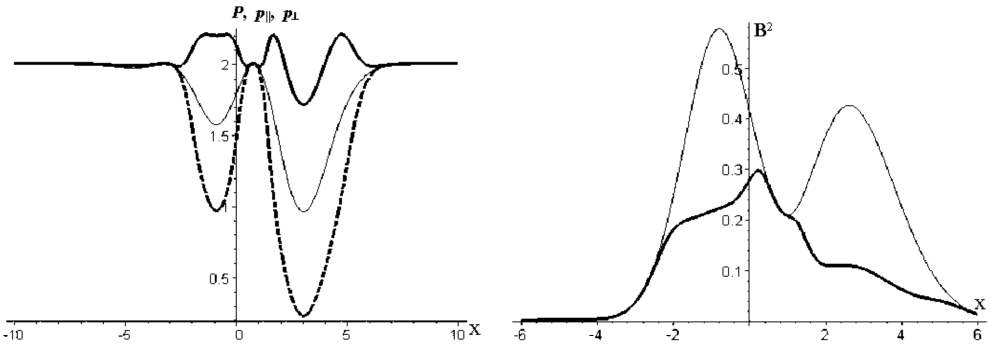
$$\psi_{110}(r, z, \phi) = \mathrm{e}^{-\beta r^2}(1 - 4\beta r^2 + a_1 r\cos(z/\gamma - \phi)). \tag{3.3}$$

Figure 1 shows the section $z = 0$ of the surfaces of its constant level: $\psi_{110}(r, z, \phi) = \text{const}$ for $a_1 = -1, \beta = 0.1, \gamma = \sqrt{5/2}, \alpha = 3/(2\gamma)$.

We now apply the 'anisotropizing' transformations (corollary 1) to the static exact isotropic solutions (3.1), (3.2), and obtain new static anisotropic equilibria (2.16)

$$\begin{aligned} \mathbf{B}_a &= f(\mathbf{r})\mathbf{B}_h \\ p_{\perp a} &= C_0\mu P_h + C_1 + (C_0 - f^2(\mathbf{r})/\mu)\mathbf{B}_h^2/2 \\ p_{\parallel a} &= C_0\mu P_h + C_1 - (C_0 - f^2(\mathbf{r})/\mu)\mathbf{B}_h^2/2 \end{aligned} \tag{3.4}$$

and $f(\mathbf{r})$ is an arbitrary function constant on the magnetic field lines; $C_0 > 0$, C_1 are arbitrary constants.



(a) (b)

Figure 2. Comparison of pressure and magnetic field profiles in isotropic and anisotropic astrophysical jet models. (a) The profiles of pressure along the x-axis for the astrophysical jet model ($a_1 = -1$, $\beta = 0.1, \gamma = \sqrt{5/2}, \alpha = 3/(2\gamma)$). Original isotropic pressure P: thin solid line, anisotropic $p_{\|a}$: thick dash line; $p_{\perp a}$: thick solid line. Positive-pressure requirement is satisfied. (b) The magnetic field magnitudes $\mathbf{B}^2$ and $\mathbf{B}_a^2$ for isotropic (thin line) and anisotropic (thick line) astrophysical jet model (the profile along the x-axis). Same parameters as in part (a).

Let us consider particular solutions from the family (3.4) in more detail.

(1) *Anisotropic helically symmetric jets.* We take the flux function in the simplest form $\psi = \psi_{110}(r, z, \phi)$, and choose a helically symmetric arbitrary function

$$f(\mathbf{r}) = (C_0 + 1/\cosh(\psi^2))^{1/2}. \tag{3.5}$$

The magnetic field and pressure functions are given by expressions (3.4).

Figure 2(a) represents the profiles of pressure along the x-axis (original isotropic pressure P_h shown with a thin solid line, anisotropic $p_{\|a}$ with a thick dash line and $p_{\perp a}$ with a thick solid line). Positive-pressure requirement is evidently satisfied.

In figure 2(b), the original isotropic and the transformed anisotropic magnetic field magnitudes $\mathbf{B}_h^2$ and $\mathbf{B}_a{}^2$ along the x-axis are shown (isotropic with a thin solid line and anisotropic with a thick solid line). The magnetic field is evidently bounded from above, therefore, in accordance with stability considerations presented in section 2.3, the presented sample solution is free from fire-hose and mirror instabilities.

Both figures use values $C_0 = 1.0, C_1 = 0.01$.

(2) *Astrophysical jet model with no symmetries.* The arbitrary function $f(\mathbf{r})$ has only to be constant on magnetic field lines, not necessarily on magnetic surfaces $\psi = \text{const}$ (cf corollary 1). In the family of solutions (3.4), the magnetic field lines all go to infinity in the variable z [12]. Therefore, the function $f(\mathbf{r})$ in this anisotropic solution depends on two transversal variables, and the generic exact solutions (3.4) are *non-symmetric*.

As an example the flux function $\psi = \psi_{110}(r, z, \phi)$ (3.3) and constants $a_1 = 0, \beta = 0.1, \gamma = \sqrt{5/2}, \alpha = 3/(2\gamma)$ (this choice indeed yields a cylindrically symmetric flux function). A simple computation shows that the general function of two variables

$$f(\mathbf{r}) = F\left(r, \phi - \frac{2\sqrt{10}z}{2r^2 - 15}\right) \tag{3.6}$$

is constant on the lines of the magnetic field (3.1). Every magnetic field line winding on a cylindrical surface $\psi = \text{const}$ goes to infinity and is helically symmetric, but the helical constant changes from line to line; therefore, the general anisotropic static plasma equilibrium solution (3.4), (3.6) has no geometrical symmetries.

4. Infinite-dimensional symmetries for anisotropic (CGL) plasma equilibria

4.1. The symmetry transforms

Recently Bogoyavlenskij [4, 5] found that the isotropic MHD equilibrium equations (2.1)–(2.3) possess the following symmetries.

If $\{\mathbf{V}(\mathbf{r}), \mathbf{B}(\mathbf{r}), P(\mathbf{r}), \rho(\mathbf{r})\}$ is an MHD equilibrium, where the density $\rho(\mathbf{r})$ is constant on both magnetic field lines and streamlines, then $\{\mathbf{V}_1(\mathbf{r}), \mathbf{B}_1(\mathbf{r}), P_1(\mathbf{r}), \rho_1(\mathbf{r})\}$ is also an equilibrium solution, where

$$\begin{aligned} \mathbf{V}_1 &= \frac{b(\mathbf{r})}{m(\mathbf{r})\sqrt{\mu\rho}}\mathbf{B} + \frac{a(\mathbf{r})}{m(\mathbf{r})}\mathbf{V} \\ \mathbf{B}_1 &= a(\mathbf{r})\mathbf{B} + b(\mathbf{r})\sqrt{\mu\rho}\mathbf{V} \\ \rho_1 &= m^2(\mathbf{r})\rho \qquad P_1 = CP + \left(C\mathbf{B}^2 - \mathbf{B}_1^2\right)/(2\mu). \end{aligned} \tag{4.1}$$

Here

$$a^2(\mathbf{r}) - b^2(\mathbf{r}) = C = \text{const}$$

and $a(\mathbf{r}), b(\mathbf{r}), c(\mathbf{r})$ are functions constant on both magnetic field lines and streamlines (i.e. on magnetic surfaces $\Psi = \text{const}$, when they exist).

These symmetries form an infinite-dimensional Abelian group [5]

$$G_m = A_m \oplus A_m \oplus R^+ \oplus Z_2 \oplus Z_2 \oplus Z_2 \tag{4.2}$$

where R^+ is a multiplicative group of positive numbers, and A_m is an additive Abelian group of smooth functions in $\mathbb{R}^3$ that are constant on magnetic surfaces. The group G_m has eight connected components.

Below we present the symmetries of ideal anisotropic (CGL) plasma equilibria (2.10), (2.11), which naturally generalize the above isotropic-case symmetries.

From now on we consider only plasma configurations free of the fire-hose instability, i.e. for which $1/\mu > \tau$ in the whole domain (see section 2).

Theorem 2. *Let* $\{\mathbf{V}(\mathbf{r}), \mathbf{B}(\mathbf{r}), p_\perp(\mathbf{r}), p_\parallel(\mathbf{r}), \rho(\mathbf{r})\}$ *be a solution of the CGL equilibrium system (2.10), (2.11), where the density* $\rho(\mathbf{r})$ *and the anisotropy factor* $\tau(\mathbf{r})$ *(2.9) are constant on both magnetic field lines and streamlines. Then* $\{\mathbf{V}_1(\mathbf{r}), \mathbf{B}_1(\mathbf{r}), p_{\perp 1}(\mathbf{r}), p_{\parallel 1}(\mathbf{r}), \rho_1(\mathbf{r})\}$ *is also a solution, where*

$$\begin{aligned} \rho_1 &= m^2(\mathbf{r})\rho \\ \mathbf{V}_1 &= \frac{b(\mathbf{r})\sqrt{1/\mu - \tau}}{m(\mathbf{r})\sqrt{\rho}}\mathbf{B} + \frac{a(\mathbf{r})}{m(\mathbf{r})}\mathbf{V} \\ \mathbf{B}_1 &= \frac{a(\mathbf{r})}{n(\mathbf{r})}\mathbf{B} + \frac{b(\mathbf{r})\sqrt{\rho}}{n(\mathbf{r})\sqrt{1/\mu - \tau}}\mathbf{V} \\ p_{\perp 1} &= Cp_\perp + \frac{\left(C\mathbf{B}^2 - \mathbf{B}_1^2\right)}{2\mu} \\ p_{\parallel 1} &= p_\parallel n^2(\mathbf{r})\frac{\mathbf{B}_1^2}{\mathbf{B}^2} + p_\perp\left(C - n^2(\mathbf{r})\frac{\mathbf{B}_1^2}{\mathbf{B}^2}\right) + \frac{\left(C\mathbf{B}^2 + {\mathbf{B}_1}^2(1 - 2n^2(\mathbf{r}))\right)}{2\mu}. \end{aligned} \tag{4.3}$$

Here

$$a^2(\mathbf{r}) - b^2(\mathbf{r}) = C = \text{const}$$

and $a(\mathbf{r}), b(\mathbf{r}), m(\mathbf{r}), n(\mathbf{r})$ *are functions constant on both magnetic field lines and streamlines.*

Under the conditions of the theorem, the anisotropy factor $\tau(\mathbf{r})$ is transformed as follows:

$$\tau_1 \equiv \frac{p_{\|1} - p_{\perp 1}}{\mathbf{B}_1{}^2} = \frac{1}{\mu} - n^2(\mathbf{r})\left(\frac{1}{\mu} - \tau\right).$$

The proof is given in appendix B.

The above symmetry transforms are applicable to any dynamic or static anisotropic CGL plasma configuration with density $\rho(\mathbf{r})$ and the anisotropy factor $\tau(\mathbf{r})$ constant on magnetic field lines and streamlines. For example, it can be directly applied to *static* anisotropic configurations that were obtained in section 3 to produce families of dynamic solutions.

Like the isotropic-case symmetries (4.1), the transformations (4.3) are invertible for $C \neq 0$:

$$C\mathbf{V} = \frac{a(\mathbf{r})}{m_1(\mathbf{r})}\mathbf{V}_1 - \frac{b(\mathbf{r})\sqrt{1/\mu - \tau_1(\mathbf{r})}}{m_1(\mathbf{r})\sqrt{\rho_1(\mathbf{r})}}\mathbf{B}_1 \qquad C\mathbf{B} = \frac{a(\mathbf{r})}{n_1(\mathbf{r})}\mathbf{B}_1 - \frac{b(\mathbf{r})\sqrt{\rho_1(\mathbf{r})}}{n_1(\mathbf{r})\sqrt{1/\mu - \tau_1(\mathbf{r})}}\mathbf{V}_1.$$

4.2. The structure of the arbitrary functions

The arbitrary functions $a(\mathbf{r}), b(\mathbf{r}), m(\mathbf{r}), n(\mathbf{r})$ must be constant on magnetic field lines and plasma streamlines, and therefore their structure depends on the topology of the original anisotropic MHD equilibrium configuration $\{\mathbf{V}(\mathbf{r}), \mathbf{B}(\mathbf{r}), \tau(\mathbf{r}), p_\perp(\mathbf{r}), \rho(\mathbf{r})\}$.

In the following topologies the structure of the unknown functions is evident:

(i) If the magnetic field $\mathbf{B}$ and velocity $\mathbf{V}$ of the original anisotropic MHD equilibrium configuration are in every point tangent to magnetic surfaces, then the functions $a(\mathbf{r}), b(\mathbf{r}), m(\mathbf{r}), n(\mathbf{r})$ must be constant on every such surface.

(ii) The magnetic field and velocity are collinear, and the field lines are closed loops or go to infinity. Then the functions $a(\mathbf{r}), b(\mathbf{r}), m(\mathbf{r}), n(\mathbf{r})$ have to be constant on the plasma streamlines.

(iii) The magnetic field and velocity are collinear, and their field lines are dense in some 3D domain $\mathcal{D}$. Then the functions $a(\mathbf{r}), b(\mathbf{r}), m(\mathbf{r}), n(\mathbf{r})$ are constant in $\mathcal{D}$.

4.3. The group structure of the symmetry transforms

Consider the set G of all transformations (4.3) with $C \neq 0$ and smooth $a(\mathbf{x}), b(\mathbf{x}), m(\mathbf{x})$ and $n(\mathbf{x})$ constant on magnetic field lines and plasma streamlines, for a given anisotropic MHD equilibrium. Each transformation is prescribed by a quadruple of functions (a, b, m, n) that satisfy the conditions

$$a^2(\mathbf{r}) - b^2(\mathbf{r}) \equiv \text{const} = C \neq 0 \qquad m(\mathbf{r}) \neq 0 \qquad n(\mathbf{r}) \neq 0.$$

The domain E for these transformations consists of all divergence-free incompressible CGL equilibria that have the anisotropy factor τ constant on magnetic surfaces.

Consider a function $h(\mathbf{r})$ constant on the lines of $\mathbf{V}$ and $\mathbf{B}$ of an initial equilibrium configuration. This implies

$$\mathbf{B} \cdot \operatorname{grad} h(\mathbf{r}) = 0 \qquad \mathbf{V} \cdot \operatorname{grad} h(\mathbf{r}) = 0.$$

For the transformed 'mixed' vector fields $\mathbf{V}_1$ and $\mathbf{B}_1$ (4.3) one also has

$$\mathbf{B}_1 \cdot \operatorname{grad} h(\mathbf{r}) = 0 \qquad \mathbf{V}_1 \cdot \operatorname{grad} h(\mathbf{r}) = 0.$$

Therefore the function $h(\mathbf{r})$ is also constant on the magnetic field lines and plasma streamlines of the new plasma equilibrium configuration. This fact, together with the invertibility of the transformations (4.3) for $C \neq 0$, proves that the range of these transformations is the same as their domain. Hence the composition of the transformations is well defined.

We now show that the composition assigns on the set G the structure of an Abelian group. Indeed, the composition of the transformations (4.3) is equivalent to the 4×4 matrix multiplication

$$\begin{pmatrix} m_2 & 0 & 0 & 0 \\ 0 & n_2 & 0 & 0 \\ 0 & 0 & a_2\sqrt{\frac{\rho_1}{\rho_2}} & b_2\frac{\sqrt{1/\mu-\tau_1}}{\sqrt{\rho_2}} \\ 0 & 0 & b_2\frac{\sqrt{\rho_1}}{\sqrt{1/\mu-\tau_2}} & a_2\frac{\sqrt{1/\mu-\tau_1}}{\sqrt{1/\mu-\tau_2}} \end{pmatrix} \times \begin{pmatrix} m_1 & 0 & 0 & 0 \\ 0 & n_1 & 0 & 0 \\ 0 & 0 & a_1\sqrt{\frac{\rho}{\rho_1}} & b_1\frac{\sqrt{1/\mu-\tau}}{\sqrt{\rho_1}} \\ 0 & 0 & b_1\frac{\sqrt{\rho}}{\sqrt{1/\mu-\tau_1}} & a_1\frac{\sqrt{1/\mu-\tau}}{\sqrt{1/\mu-\tau_1}} \end{pmatrix}$$

$$= \begin{pmatrix} m & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & a\sqrt{\frac{\rho}{\rho_2}} & b\frac{\sqrt{1/\mu-\tau}}{\sqrt{\rho_2}} \\ 0 & 0 & b\frac{\sqrt{\rho}}{\sqrt{1/\mu-\tau_2}} & a\frac{\sqrt{1/\mu-\tau}}{\sqrt{1/\mu-\tau_2}} \end{pmatrix}$$

where $m = m_2m_1, n = n_2n_1, a = a_2a_1 + b_2b_1, b = b_2a_1 + a_2b_1$. In other words,

$$(m_2, n_2, a_2, b_2) \cdot (m_1, n_1, a_1, b_1) = (m_2m_1, n_2n_1, a_2a_1 + b_2b_1, b_2a_1 + a_2b_1) \tag{4.4}$$

which implies $C = a^2 - b^2 = C_2C_1 \neq 0$. The unit quadruple is $(1, 1, 1, 0)$, and the inverse transform corresponds to the quadruple $(m, n, a, b)^{-1} = (m^{-1}, n^{-1}, C^{-1}a, -C^{-1}b)$. It is evident that the above multiplication is commutative and associative. Hence the symmetries (4.3) form an Abelian group G.

Let us describe the structure of the group G. We introduce the parametrization $m(\mathbf{x}) = \theta \exp\alpha(\mathbf{r}), n(\mathbf{x}) = \lambda \exp\beta(\mathbf{r})$, where $\alpha(\mathbf{r}), \beta(\mathbf{r})$ are smooth functions constant on the magnetic field lines and plasma streamlines; $\theta, \lambda = \pm 1$. For $C = \sigma k^2, \sigma = \pm 1, k > 0$, the equation $a^2(\mathbf{r}) - b^2(\mathbf{r}) = C$ is resolved in the form $\sigma = 1$: $a(\mathbf{r}) = \eta k \operatorname{ch}\delta(\mathbf{r}), b(\mathbf{r}) = \eta k \operatorname{sh}\delta(\mathbf{r})$; $\sigma = -1$: $a(\mathbf{r}) = \eta k \operatorname{sh}\delta(\mathbf{r}), b(\mathbf{x}) = \eta k \operatorname{ch}\delta(\mathbf{r})$, where $\eta = \pm 1$ and $\delta(\mathbf{x})$ is an arbitrary smooth function constant on the magnetic field lines and plasma streamlines. Hence each transformation (4.3) corresponds to an octuple $(\alpha(\mathbf{r}), \beta(\mathbf{r}), \delta(\mathbf{r}), k, \theta, \lambda, \sigma, \eta)$. The group multiplication law then can be written in the form

$$\begin{aligned}(\alpha_1(\mathbf{r}), \beta_1(\mathbf{r}), \delta_1(\mathbf{r}), k_1, \theta_1, \lambda_1, \sigma_1, \eta_1) \cdot (\alpha_2(\mathbf{x}), \beta_2(\mathbf{x}), \delta_2(\mathbf{r}), k_2, \theta_2, \lambda_2, \sigma_2, \eta_2) \\ = (\alpha_1(\mathbf{r}) + \alpha_2(\mathbf{r}), \beta_1(\mathbf{r}) + \beta_2(\mathbf{r}), \delta_1(\mathbf{r}) + \delta_2(\mathbf{r}), k_1k_2, \theta_1\theta_2, \lambda_1\lambda_2, \sigma_1\sigma_2, \eta_1\eta_2).\end{aligned}$$

Hence the group G is the direct sum

$$G = A_m \oplus A_m \oplus A_m \oplus R^+ \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2. \tag{4.5}$$

Here R^+ is the multiplicative group of positive numbers $k > 0$. The A_m is the additive Abelian group of smooth functions in $\mathbb{R}^3$ that are constant on the magnetic field lines and plasma streamlines for a given anisotropic MHD equilibrium. The group A_m is a linear space and an associative algebra with respect to the multiplication of functions. The group G evidently has 16 connected components.

The group G_m of the isotropic-case transformations (4.1), (4.2) constitutes an infinite-dimensional subgroup of G.

The transformations (4.3) are a direct generalization of the symmetries (4.1) of the isotropic equilibrium equations (2.1), (2.2). The isotropic-case and anisotropic-case transformations coincide for the choice $\tau = 0, n(\mathbf{r}) = 1$.

Remark 5. If the original plasma configuration possesses the inequality $1/\mu - \tau > 0$, and thus is free of the fire-hose instability, then the transformed configuration is also fire-hose stable when $1/\mu - \tau_1 = n^2(\mathbf{r})(1/\mu - \tau) > 0$. The latter is always true if $n(\mathbf{r}) \neq 0$ in the plasma domain. Therefore the symmetries (4.3) do not produce the fire-hose instability.

4.4. Connections with the Lie point transformations

It was shown in [13] that the symmetries (4.1) of the isotropic plasma equilibrium system are equivalent to certain Lie point transformations of that system, and can be obtained directly from the Lie group analysis procedure, provided that the general solution topology (the existence of magnetic surfaces to which vector fields **B** and **V** are tangent) and the incompressibility condition are explicitly taken into account in the form of additional constraints:

$$\rho(\mathbf{r}) = \rho(\Psi(\mathbf{r})) \qquad \text{grad}(\Psi(\mathbf{r}))\cdot\mathbf{B} = 0 \qquad \text{grad}(\Psi(\mathbf{r}))\cdot\mathbf{V} = 0.$$

Here $\Psi(\mathbf{r})$ is a magnetic surface function (or, more generally, a function constant on magnetic field lines and plasma streamlines.) In [14], it is proved that the symmetries of the *anisotropic* plasma equilibria also are equivalent to some special infinite-dimensional Lie point transformations.

5. Conclusions

In this paper we introduce two methods of constructing new anisotropic plasma equilibrium configurations satisfying the Chew–Goldberger–Low equations (2.10), (2.11) [2].

In section 2 we present infinite-dimensional transformations between isotropic (MHD) and anisotropic (CGL) plasma equilibria. These transformations can be applied to any static plasma equilibrium and to a wide class of dynamic equilibria (those with density ρ constant on plasma streamlines and magnetic field lines). Unlike the Bäcklund transforms, the presented ones are expressed in the explicit form and depend on three spatial variables. The resulting anisotropic solutions retain the topology of the original isotropic plasma equilibrium.

In section 2.2 we discuss the form of the transformations when they are applied to static MHD equilibria ($\mathbf{V} = \mathbf{0}$). It appears that the transformations can be applied even to degenerate plasma equilibria—pure magnetic fields in vacuum—to produce non-degenerate CGL plasma equilibria.

The boundary conditions and stability of the equilibrium configurations are studied in section 2.3. We show that the anisotropic solutions obtained by applying the transformations (2.12) can be made free of fire-hose and mirror instability by the proper choice of transformation parameters.

In section 3 we construct exact CGL equilibria modelling anisotropic astrophysical jets. It is based on the family of isotropic MHD equilibria derived in [12] and includes solutions with no geometrical symmetries.

In section 4 we introduce a family of topology-dependent infinite-dimensional symmetries (4.3) of anisotropic (CGL) incompressible plasma equilibrium equations. These symmetries can be used to produce families of CGL equilibrium solutions in an explicit algebraic form. They depend on three arbitrary functions that are constant on magnetic field lines and plasma streamlines of the original anisotropic equilibrium. The transformations form an Abelian group $G = A_m \oplus A_m \oplus A_m \oplus R^{+} \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2$ (4.5) with 16 connected components.

The presented symmetries generalize the known symmetries (4.1) for the isotropic incompressible MHD equilibria. The group $G_m = A_m \oplus A_m \oplus R^{+} \oplus Z_2 \oplus Z_2 \oplus Z_2$ (4.2) is a subgroup of the group G (4.5).

The symmetries (4.3) depend on all three spatial variables $\mathbf{r} = (x, y, z)$ and break geometrical symmetries, if the original equilibrium is field aligned and the field lines either are closed curves or go to infinity.

Using these symmetries, one can construct dynamic CGL plasma equilibria from static ones. The symmetries are shown to create solutions that are free from the fire-hose instability.

Similar to the symmetries of isotropic plasma equilibria, the anisotropic equilibrium symmetries (4.3) are equivalent to certain Lie point transformations of the CGL equilibrium system. The corresponding Lie point symmetries can be found by the general Lie group analysis of the CGL equilibrium system only if the existence of magnetic surfaces and the incompressibility condition are explicitly taken into account.

Appendix A. Proof of theorem 1

Let us insert the quantities (2.12) into the system of CGL plasma equilibrium equations (2.10), (2.11), assuming that $\{\mathbf{V}(\mathbf{r}), \mathbf{B}(\mathbf{r}), P(\mathbf{r}), \rho(\mathbf{r})\}$ is an anisotropic MHD equilibrium and satisfies (2.1)–(2.3).

To simplify the notation, we do not write the dependence of functions on $\mathbf{r}$ explicitly.

The functions $f(\mathbf{r})$, $g(\mathbf{r})$ are constant on the magnetic field lines and streamlines, therefore

$$\operatorname{div}\mathbf{B}_1 = f\operatorname{div}\mathbf{B} + \mathbf{B}\operatorname{grad} f = 0 \qquad \operatorname{div}\mathbf{V}_1 = g\operatorname{div}\mathbf{V} + \mathbf{V}\operatorname{grad} g = 0 \tag{A.1}$$

Also, using a vector calculus identity

$$\operatorname{curl}(s\mathbf{q}) = s\operatorname{curl}\mathbf{q} + \operatorname{grad}(s)\times\mathbf{q} \tag{A.2}$$

we conclude that

$$\operatorname{curl}(\mathbf{V}_1\times\mathbf{B}_1) = 0 \tag{A.3}$$

therefore equations (2.11) are satisfied.

To prove that (2.10) holds, we first observe that

$$\begin{aligned}
\rho_1\mathbf{V}_1\times\operatorname{curl}\mathbf{V}_1 - \left(\frac{1}{\mu}-\tau_1\right)\mathbf{B}_1\times\operatorname{curl}\mathbf{B}_1 &= \rho_1 g^2\mathbf{V}\times\operatorname{curl}\mathbf{V} - \left(\frac{1}{\mu}-\tau_1\right)f^2\mathbf{B}\times\operatorname{curl}\mathbf{B}\\
&\quad + \mathbf{V}^2\rho_1 g\operatorname{grad}(g) - \mathbf{B}^2\left(\frac{1}{\mu}-\tau_1\right)f\operatorname{grad}(f)\\
&= C_0\mu\left(\rho\mathbf{V}\times\operatorname{curl}\mathbf{V} - \frac{1}{\mu}\mathbf{B}\times\operatorname{curl}\mathbf{B}\right) + \mathbf{V}^2\rho_1 g\operatorname{grad}(g) - \mathbf{B}^2\left(\frac{1}{\mu}-\tau_1\right)f\operatorname{grad}(f)\\
&= C_0\mu\left(\operatorname{grad}P + \rho\operatorname{grad}\frac{\mathbf{V}^2}{2}\right) + \mathbf{V}^2\rho_1 g\operatorname{grad}(g) - \mathbf{B}^2\left(\frac{1}{\mu}-\tau_1\right)f\operatorname{grad}(f)\\
&= C_0\mu\operatorname{grad}P + C_0\rho\mu\operatorname{grad}\mathbf{V}^2/2 + \frac{C_0\rho\mu\mathbf{V}^2}{2g^2}\operatorname{grad}g^2 - \frac{\mathbf{B}_1{}^2}{2}\operatorname{grad}(\tau_1).
\end{aligned}$$

According to the remark (2.13), τ_1 is constant on both magnetic field lines and streamlines, therefore

$$\mathbf{B}_1\cdot\operatorname{grad}\tau_1 = 0.$$

The right-hand side of (2.10) is

$$\begin{aligned}
\operatorname{grad}p_{\perp 1} + \rho_1\operatorname{grad}\frac{\mathbf{V}_1{}^2}{2} + \tau_1\operatorname{grad}\frac{\mathbf{B}_1{}^2}{2} &= \operatorname{grad}\left(p_{\perp 1} + \rho_1\frac{\mathbf{V}_1{}^2}{2} + \tau_1\frac{\mathbf{B}_1{}^2}{2}\right)\\
&\quad - \frac{\mathbf{B}_1{}^2}{2}\operatorname{grad}(\tau_1) - \frac{\mathbf{V}_1{}^2}{2}\operatorname{grad}(\rho_1) = C_0\mu\operatorname{grad}P + C_0\rho\mu\operatorname{grad}\mathbf{V}^2/2\\
&\quad + \frac{C_0\rho\mu\mathbf{V}^2}{2g^2}\operatorname{grad}g^2 - \frac{\mathbf{B}_1{}^2}{2}\operatorname{grad}(\tau_1)
\end{aligned}$$

and is identically equal to the left-hand side. Hence the theorem is proved.

Appendix B. Proof of theorem 2

First, we remark that from pressure transformation formulae (4.3) it follows that

$$\tau_1 \equiv \frac{p_{\parallel 1} - p_{\perp 1}}{\mathbf{B}_1{}^2} = \frac{1}{\mu} - n^2(\mathbf{r}) \left(\frac{1}{\mu} - \tau \right).$$

Now let $w(\mathbf{x})$ be any function that is constant on magnetic field lines and plasma streamlines. Then

$$\mathbf{B} \cdot \operatorname{grad} w(\mathbf{x}) = 0 \qquad \mathbf{V} \cdot \operatorname{grad} w(\mathbf{x}) = 0. \tag{B.1}$$

For any smooth vector field $\mathbf{A}$, a vector calculus identity holds

$$\mathbf{A} \times \operatorname{curl} \mathbf{A} = -(\mathbf{A} \cdot \operatorname{grad})\mathbf{A} + \operatorname{grad}(\mathbf{A}^2/2). \tag{B.2}$$

Using it, we rewrite equation (2.10) under consideration, assuming the density $\rho(\mathbf{r})$ and the anisotropy factor $\tau(\mathbf{r})$ (2.9) constant on both magnetic field lines and streamlines, as

$$\rho(\mathbf{V} \cdot \operatorname{grad})\mathbf{V} - \left(\frac{1}{\mu} - \tau \right) (\mathbf{B} \cdot \operatorname{grad})\mathbf{B} = -\operatorname{grad} \left(p_\perp + \frac{\mathbf{B}^2}{2\mu} \right). \tag{B.3}$$

In (4.3), the coefficients at $\mathbf{B}$, $\mathbf{V}$ in the formulae defining $\mathbf{B}_1$, $\mathbf{V}_1$ are evidently constant on magnetic field lines and plasma streamlines.

Using formulae (B.1), (B.2), we get

$$\begin{aligned}\rho(\mathbf{V} \cdot \operatorname{grad})\mathbf{V} - \left(\frac{1}{\mu} - \tau \right) (\mathbf{B} \cdot \operatorname{grad})\mathbf{B} + \operatorname{grad} \left(p_\perp + \frac{\mathbf{B}^2}{2\mu} \right) = (a^2(\mathbf{r}) - b^2(\mathbf{r})) \\ \times \left(\rho(\mathbf{V}_1 \cdot \operatorname{grad})\mathbf{V}_1 - \left(\frac{1}{\mu} - \tau_1 \right) (\mathbf{B}_1 \cdot \operatorname{grad})\mathbf{B}_1 + \operatorname{grad} \left(p_{\perp 1} + \frac{\mathbf{B}_1{}^2}{2\mu} \right) \right).\end{aligned} \tag{B.4}$$

Thus the functions $\rho_1, \mathbf{B}_1, \mathbf{V}_1, p_{\perp 1}, p_{\parallel 1}$ satisfy equation (B.3) and therefore the CGL equilibrium equation (2.10).

The equations $\operatorname{div} \mathbf{V}_1 = 0$, $\operatorname{div} \mathbf{B}_1 = 0$ are evidently satisfied due to (B.1).

Now consider the quantity

$$\mathbf{V}_1 \times \mathbf{B}_1 = \frac{a^2(\mathbf{r}) - b^2(\mathbf{r})}{m(\mathbf{r})n(\mathbf{r})} (\mathbf{V} \times \mathbf{B}).$$

The scalar factor on the left-hand side is constant on magnetic field lines and plasma streamlines, therefore (B.1) applies. Hence

$$\operatorname{curl}(\mathbf{V}_1 \times \mathbf{B}_1) = \frac{a^2(\mathbf{r}) - b^2(\mathbf{r})}{m(\mathbf{r})n(\mathbf{r})} \operatorname{curl}(\mathbf{V} \times \mathbf{B}) + \operatorname{grad} \left(\frac{a^2(\mathbf{r}) - b^2(\mathbf{r})}{m(\mathbf{r})n(\mathbf{r})} \right) \cdot (\mathbf{V} \times \mathbf{B}) = 0.$$

Thus all anisotropic CGL equilibrium equations (2.10), (2.11) are satisfied by the transformed quantities (4.3), and the theorem is proved.

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