On Symmetries of Linear and Nonlinear PDEs

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Outline

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Model equations

- A given system of PDEs of order k:

$$R^\sigma[\mathbf{u}] = R^\sigma(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \ldots, \partial^k\mathbf{u}) = 0, \quad \sigma = 1, \ldots, N.$$

- Higher-order symmetries are rather uncommon (though important). We will talk about point symmetries in this lecture.

Point symmetries

- A one-parameter Lie group of point transformations preserving the model:

$$\begin{aligned}(x^i)^* &= f^i(\mathbf{x}, \mathbf{u}; \varepsilon) = x^i + \varepsilon\xi^i(\mathbf{x}, \mathbf{u}) + O(\varepsilon^2),\\ (u^\mu)^* &= g^\mu(\mathbf{x}, \mathbf{u}; \varepsilon) = u^\mu + \varepsilon\eta^\mu(\mathbf{x}, \mathbf{u}) + O(\varepsilon^2).\end{aligned}$$

- The corresponding infinitesimal generator (tangent vector field):

$$\mathrm{X} = \xi^i(\mathbf{x}, \mathbf{u})\frac{\partial}{\partial x^i} + \eta^\mu(\mathbf{x}, \mathbf{u})\frac{\partial}{\partial u^\mu}.$$

- Finite-dimensional group – general case for nonlinear PDEs.
- Infinite-dimensional group (parameterized by arbitrary function(s) with of fewer arguments than # of independent variables) – occurs for nonlinear models (e.g., Galilei group).
- Infinite-dimensional group (parameterized by arbitrary function(s), # arguments = # independent variables) – common for linear PDEs, and PDEs that may be linearized by a point transformation.

Outline

- If the given PDE system

$$R^\sigma[\mathbf{u}] = R^\sigma(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \ldots, \partial^k\mathbf{u}) = 0, \quad \sigma = 1, \ldots, N$$

is linear, then the evolutionary symmetry components are arbitrary solutions of the linearized equations (linear homogeneous PDEs)

$$\mathcal{L}\{R\}^\sigma_\mu[\mathbf{u}]\zeta^\mu = 0, \quad \sigma = 1, \ldots, N.$$

- Example: $u = u(x, t)$,

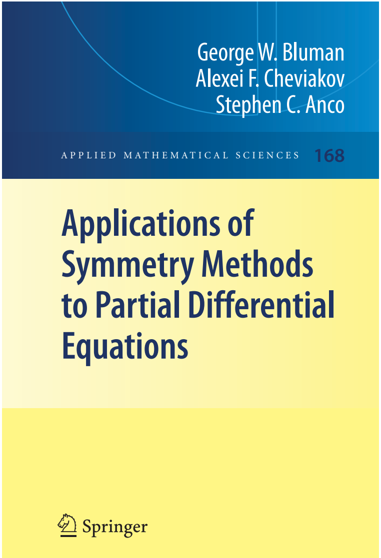
$$u_t = u_{xx},$$

has an infinite point symmetry group with $\mathrm{X} = g(x, t)\dfrac{\partial}{\partial u}$,

$$x^* = x, \quad t^* = t, \quad u^* = u + g(x, t),$$

where $g_t = g_{xx}$.

Outline


George W. Bluman
Alexei F. Cheviakov
Stephen C. Anco
APPLIED MATHEMATICAL SCIENCES 168
Applications of Symmetry Methods to Partial Differential Equations
Springer

Theorem 2.4.1 (Necessary conditions for the existence of an invertible linearization mapping of a nonlinear PDE system)**.** *If there exists an invertible mapping μ of a given nonlinear PDE system $\mathbf{R}\{x\,;u\}$ $(m \geq 2)$ to some linear PDE system $\mathbf{S}\{z\,;w\}$, then*

(i) μ is a point transformation of the form

$$z^j = \phi^j(x,u), \quad j = 1,\dots,n, \tag{2.61a}$$

$$w^\gamma = \psi^\gamma(x,u), \quad \gamma = 1,\dots,m; \tag{2.61b}$$

(ii) $\mathbf{R}\{x\,;u\}$ has an infinite set of point symmetries given by an infinitesimal generator

$$\mathrm{X} = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\nu(x,u)\frac{\partial}{\partial u^\nu} \tag{2.62}$$

with infinitesimals $\xi_i(x,u)$, $\eta^\nu(x,u)$ of the form

$$\xi^i(x,u) = \alpha^i_\sigma(x,u)F^\sigma(x,u), \tag{2.63a}$$

$$\eta^\nu(x,u) = \beta^\nu_\sigma(x,u)F^\sigma(x,u), \tag{2.63b}$$

where $\alpha^i_\sigma(x,u)$, $\beta^\nu_\sigma(x,u)$, $i = 1,\dots,n$; $\nu;\sigma = 1,\dots,m$, are specific functions of x and u, and where $F = (F^1,\dots,F^m)$ is an arbitrary solution of some linear PDE system

$$\mathrm{L}[X]F = 0 \tag{2.64}$$

in terms of some linear differential operator $\mathrm{L}[X]$ and specific independent variables $X = (X^1(x,u),\dots,X^n(x,u)) = (\phi^1,\dots,\phi^n)$.

Infinite symmetries – linearization by a point transformation

- Some PDE systems can be linearized by a nonlocal transformation (have a sufficiently large set of nonlocal symmetries).

- E.g. Burgers equation: $u_t + uu_x - u_{xx} = 0$, $u = u(x,t)$: finitely many point/contact symmetries.

- Potential equations $v_x = 2u$, $v_t = 2u_x - u^2$ have an infinite number of point symmetries given by the infinitesimal generator

$$\mathrm{X} = e^{v/4}\left\{[2h(x,t) + g(x,t)u]\frac{\partial}{\partial u} + 4g(x,t)\frac{\partial}{\partial v}\right\},$$

  where $(g(x,t), h(x,t))$ is an arbitrary solution of the linear PDE system

$$h = g_x, \quad h_x = g_t.$$

- As a result, the Hopf-Cole transformation $u = 2y_x/y$ maps (non-invertibly) the Burgers equation into a linear diffusion equation:

$$\frac{\partial}{\partial x}\left(\frac{2}{y}(y_t - y_{xx})\right) = 0.$$

Outline

- We will mostly follow this paper:

Cheviakov, A. (2010).
Symbolic computation of local symmetries of nonlinear and linear partial and ordinary differential equations. *Mathematics in Computer Science*, 4(2-3), 203–222.

- Consider a linear DE system

$$L^\sigma(x, u, \partial u, \ldots, \partial^k u) = F^\sigma(x), \quad \sigma = 1, \ldots, N,$$

of order k, with $n \geq 2$ independent variables $x = (x^1, \ldots, x^n)$, and $m \geq 1$ dependent variables $u(x) = (u^1(x), \ldots, u^m(x))$.

- Here each $L^\sigma[u]$ is a linear homogeneous differential expression in $u(x)$.

- If $u(x)$ is a solution of the linear system $L^\sigma[u] = F^\sigma(x)$, and $w(x)$ is a solution of the linear homogeneous system $L^\sigma[w] = 0$, then

$$\hat{u}(x) = u(x) + w(x)$$

is also a solution of the linear system: $L^\sigma[\hat{u}] = F^\sigma(x)$.

- The transformation $u(x) \to u(x) + w(x)$ yields an infinite set of *trivial* Lie point symmetries

$$\mathrm{X}_{tr} = w^\mu(x) \frac{\partial}{\partial u^\mu}.$$

Symmetries of linear ODEs and PDEs

Linear ODEs

- For linear ODEs, the dimension of Lie group is always finite, which is better from the point of view of symbolic computations.
- If the explicit form of the general solution of the linear homogeneous ODE is unknown, symbolic software would not be able to compute all point symmetries explicitly.

Linear PDEs

- For linear PDEs, the dimension of point symmetry Lie algebra is infinity.
- Trivial symmetry components are general solutions of linear homogeneous PDE(s).
- These PDEs do not have closed-form solutions, so symbolic software is not able to compute these symmetries explicitly.

Linear DEs: symmetry structure theorems

- The following helpful theorems has been established in this paper:

Bluman, G. (1990).
Simplifying the form of Lie groups admitted by a given differential equation. *Journal of mathematical analysis and applications*, 145(1), 52-62.

Theorem

Suppose $L[u] = F(x)$ *is a scalar linear PDE (i.e.,* $N = m = 1, n \geq 2$*) of order* $k \geq 2$*. Then components* ξ^i, η *of its point symmetries satisfy*

$$\frac{\partial \xi^i}{\partial u} = \frac{\partial^2 \eta}{\partial u^2} = 0, \quad i = 1, \ldots, n.$$

Theorem

Suppose $L[u] = F(x)$ *is a scalar linear ODE (i.e.,* $N = m = n = 1$*) of order* $k \geq 3$*. Then components* ξ, η *of its point symmetries satisfy*

$$\frac{\partial \xi}{\partial u} = \frac{\partial^2 \eta}{\partial u^2} = 0.$$

Ovsiannikov, L. V. (1982).
Group Analysis of Differential Equations, Academic Press.

Ovsiannikov's "linear DE conjecture"

For a linear DE system $L^\sigma[\mathbf{u}] = F^\sigma(\mathbf{x})$, components ξ^i, η^μ of any point symmetry

$$\mathrm{X} = \xi^i(\mathbf{x}, \mathbf{u})\frac{\partial}{\partial x^i} + \eta^\mu(\mathbf{x}, \mathbf{u})\frac{\partial}{\partial u^\mu}$$

satisfy

$$\frac{\partial \xi^i}{\partial u^\nu} = 0, \quad \frac{\partial^2 \eta^\mu}{\partial u^\nu u^\lambda} = 0, \quad i = 1, \ldots, n, \quad \mu, \nu, \lambda = 1, \ldots, m.$$

- It is stated to hold for the "majority of linear DEs" (that is, PDE and ODE systems).
- Is it true?

Conjecture verification and use

Ovsiannikov's "linear DE conjecture"

For a linear DE system $L^\sigma[\mathbf{u})] = F^\sigma(\mathbf{x})$, components ξ^i, η^μ of any point symmetry

$$\mathrm{X} = \xi^i(\mathbf{x}, \mathbf{u})\frac{\partial}{\partial x^i} + \eta^\mu(\mathbf{x}, \mathbf{u})\frac{\partial}{\partial u^\mu}$$

satisfy

$$\frac{\partial \xi^i}{\partial u^\nu} = 0, \quad \frac{\partial^2 \eta^\mu}{\partial u^\nu u^\lambda} = 0, \quad i = 1, \ldots, n, \quad \mu, \nu, \lambda = 1, \ldots, m.$$

- The conjecture can be verified symbolically: use <> capability of rifsimp.

1. Symmetries of the linear homogeneous heat equation: $u_t = u_{xx}$.

2. Symmetries of the linear non-homogeneous heat equation: $u_t = u_{xx} + f(x)$.

3. Symmetries of the linear wave equation: $u_{tt} = c^2(x) u_{xx}$.

Outline

Notation; material picture

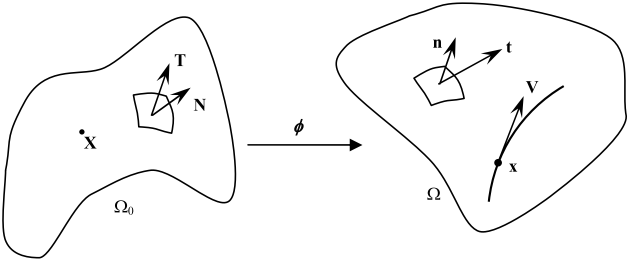


Fig. 1. Material and Eulerian coordinates.

Material picture

- A solid body occupies the reference (Lagrangian) volume $\Omega_0 \subset \mathbb{R}^3$.
- Actual (Eulerian) configuration: $\Omega \subset \mathbb{R}^3$.
- Material points are labelled by $\mathbf{X} \in \Omega_0$.
- The actual position of a material point: $\mathbf{x} = \phi(\mathbf{X}, t) \in \Omega$.

Notation; material picture

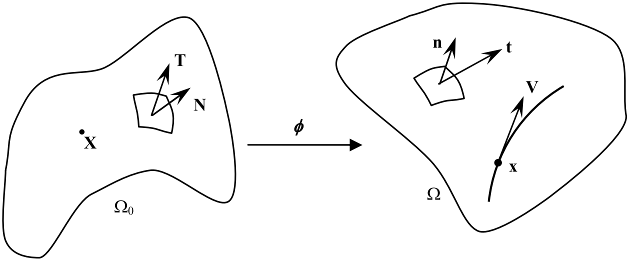


Fig. 1. Material and Eulerian coordinates.

Material picture

- Velocity of a material point $\mathbf{X}$: $\mathbf{v}(\mathbf{X}, t) = \dfrac{d\mathbf{x}}{dt}$.
- Jacobian matrix (deformation gradient):

$$\mathbf{F}(\mathbf{X}, t) = \nabla\phi; \quad J = \det \mathbf{F} > 0;$$

$$\mathbf{F} = \{F_{ij}\} = \{F^i{}_j\}.$$

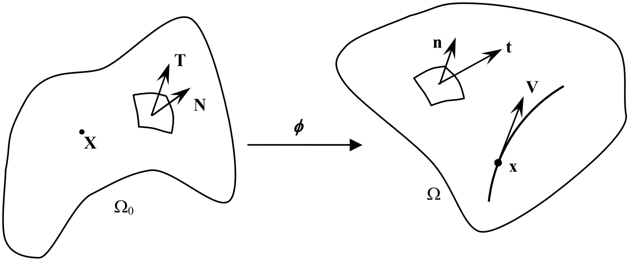


Fig. 1. Material and Eulerian coordinates.

Material picture

- Boundary force (per unit area) in Eulerian configuration: $\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}$.
- Boundary force (per unit area) in Lagrangian configuration: $\mathbf{T} = \mathbf{P}\mathbf{N}$.
- $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathrm{x}, t)$ is the Cauchy stress tensor.
- $\mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^{-T}$ is the first Piola-Kirchhoff tensor.

Notation; material picture

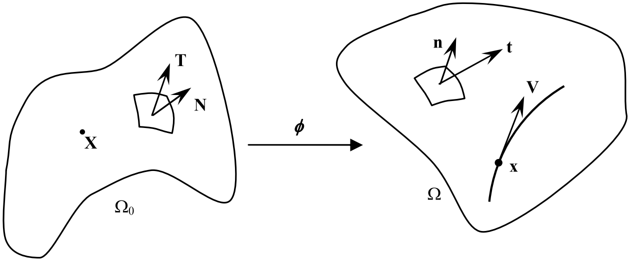


Fig. 1. Material and Eulerian coordinates.

Material picture

- Density in reference configuration: $\rho_0 = \rho_0(\mathbf{X})$ (time-independent).
- Density in actual configuration:

$$\rho = \rho(\mathbf{X}, t) = \rho_0 / J.$$

Equations of motion (no dissipation, purely elastic setting):

$$\rho_0 \mathbf{x}_{tt} = \mathrm{div}_{(X)} \mathbf{P} + \rho_0 \mathbf{R},$$

- $\mathbf{R} = \mathbf{R}(\mathbf{X}, t)$: total body force per unit mass.
- $(\mathrm{div}_{(X)} \mathbf{P})^i = \dfrac{\partial P^{ij}}{\partial X^j}$.

Cauchy stress tensor symmetry (conservation of angular momentum):

$$\mathbf{F}\mathbf{P}^T = \mathbf{P}\mathbf{F}^T \quad \Leftrightarrow \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^T.$$

The first Piola-Kirchhoff stress tensor:

$$\mathbf{P} = \rho_0 \frac{\partial W}{\partial \mathbf{F}}, \qquad P^{ij} = \rho_0 \frac{\partial W}{\partial F_{ij}}.$$

- $W = W(\mathbf{X}, \mathbf{F})$: a scalar strain energy density function.

Strain energy density for isotropic homogeneous hyperelastic materials

Isotropic homogeneous hyperelastic materials

- Strain energy density W depends only on certain matrix invariants:

$$W = U(I_1, I_2, I_3) = \bar{U}(\bar{I}_1, \bar{I}_2, \bar{I}_3).$$

- For the left Cauchy-Green strain tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^T$,

$$I_1 = \mathrm{Tr}\,\mathbf{B} = F^i{}_k F^i{}_k = \lambda_1^2 + \lambda_2^2 + \lambda_3^2,$$

$$I_2 = \tfrac{1}{2}[(\mathrm{Tr}\,\mathbf{B})^2 - \mathrm{Tr}(\mathbf{B}^2)] = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_1^2\lambda_3^2, ,$$

$$I_3 = \det \mathbf{B} = J^2 = \lambda_1^2\lambda_2^2\lambda_3^2.$$

Isotropic homogeneous hyperelastic materials

- Strain energy density W depends only on certain matrix invariants:

$$W = U(I_1, I_2, I_3) = \bar{U}(\bar{I}_1, \bar{I}_2, \bar{I}_3).$$

Table 1: Neo-Hookean and Mooney-Rivlin constitutive models

Type	Neo-Hookean	Mooney-Rivlin
Standard	$W = aI_1$, $a > 0$.	$W = aI_1 + bI_2$, $a, b > 0$
Generalized	$W = a\bar{I}_1 + c(J-1)^2$, $a, c > 0$.	$W = a\bar{I}_1 + b\bar{I}_2 + c(J-1)^2$ $a, b, c > 0$
Generalized (Ciarlet) "compressible"	$W = aI_1 + \Gamma(J)$, $\Gamma(q) = cq^2 - d\log q, \quad a, c, d > 0$	$W = aI_1 + bI_2 + \Gamma(J)$ $\Gamma(q) = cq^2 - d\log q, \quad a, b, c, d > 0$

Strain energy density for isotropic homogeneous hyperelastic materials

Isotropic homogeneous hyperelastic materials

- Strain energy density W depends only on certain matrix invariants:

$$W = U(I_1, I_2, I_3) = \bar{U}(\bar{I}_1, \bar{I}_2, \bar{I}_3).$$

Example: the Neo-Hookean Case

- Strain energy density: $W = a\, I_1, \quad a = \text{const.}$
- Equations of motion are linear and decoupled:

$$(x^k)_{tt} = a\left(\frac{\partial^2}{\partial(X^1)^2} + \frac{\partial^2}{\partial(X^2)^2} + \frac{\partial^2}{\partial(X^3)^2}\right) x^k,$$
$$k = 1, 2, 3.$$

Strain energy density for isotropic homogeneous hyperelastic materials

Isotropic homogeneous hyperelastic materials

- Strain energy density W depends only on certain matrix invariants:

$$W = U(I_1, I_2, I_3) = \bar{U}(\bar{I}_1, \bar{I}_2, \bar{I}_3).$$

General compressible framework

- Strain energy density:

$$W = \bar{U}(\bar{I}_1, \bar{I}_2, \bar{I}_3).$$

- Barred invariants:

$$\bar{I}_1 = J^{-2/3} I_1, \quad \bar{I}_2 = J^{-4/3} I_2, \quad \bar{I}_3 = J$$

- Piola-Kirchhoff stress tensor – incompressible case:

$$\mathbf{P} = -p\,\mathbf{F}^{-T} + \rho_0 \frac{\partial W}{\partial \mathbf{F}}, \qquad P^{ij} = -p\,(F^{-1})^{ji} + \rho_0 \frac{\partial W}{\partial F_{ij}}.$$

2D incompressible Mooney-Rivlin hyperelastic materials

- A 2D setting:

$$x^{1,2} = x^{1,2}\left(X^1, X^2, t\right), \qquad x^3 = X^3,$$

$$R^{1,2,3} = 0, \qquad \rho_0 = \text{const}.$$

- Derivative notation:

$$\frac{\partial^2 x^1}{\partial t^2} \equiv x^1_{tt}, \quad \frac{\partial x^1}{\partial X^2} \equiv x^1_2, \quad \frac{\partial^2 x^2}{\partial X^1 \partial X^2} \equiv x^2_{12}, \quad \frac{\partial^2 p}{\partial X^2 \partial t} \equiv p_{2\,t},$$

- Equations of motion:

$$R^1[x,p] = 1 - J = 1 - \left(x^1_1 x^2_2 - x^1_2 x^2_1\right) = 0,$$

$$R^2[x,p] = x^1_{tt} - \left[\alpha\left(x^1_{11} + x^1_{22}\right) - p_1 x^2_2 + p_2 x^2_1\right] = 0,$$

$$R^3[x,p] = x^2_{tt} - \left[\alpha\left(x^2_{11} + x^2_{22}\right) - p_2 x^1_1 + p_1 x^1_2\right] = 0,$$

where $\alpha = 2(a+b) = \text{const} > 0$ is a material parameter.

- The model has a classical Lagrangian:

$$\mathcal{L} = -K + W - p(J-1),$$

where

$$K = \frac{1}{2}\left((x_t^1)^2 + (x_t^2)^2\right)$$

is the kinetic energy density,

$$W = \frac{\alpha}{2}\left(\left(x_1^1\right)^2 + \left(x_2^1\right)^2 + \left(x_1^2\right)^2 + \left(x_2^2\right)^2\right)$$

is the and potential (strain) energy density.

- A Lagrangian density equivalent to the above (up to a total divergence) can be obtained using a homotopy formula

$$\widehat{\mathcal{L}} = \int_0^1 u \cdot R[\lambda u]\, d\lambda.$$

The Mooney-Rivlin equations arise as Euler-Lagrange equations under the actions of the Euler operators:

$$\mathrm{E}_p\,\mathcal{L} = R^1[x,p], \qquad \mathrm{E}_{x^1}\,\mathcal{L} = R^2[x,p], \qquad \mathrm{E}_{x^2}\,\mathcal{L} = R^3[x,p].$$

Extended Kovalevskaya form

- The above 2D equations admit an extended Kovalevskaya form:

$$\widehat{R}^1[x,p] = x^1_1 - \left(x^2_2\right)^{-1} S[x^1,x^2] = 0,$$

$$\widehat{R}^2[x,p] = x^2_{11} - \left(-x^2_{22} + \alpha^{-1}\left[x^2_{tt} - p_1 x^1_2 + p_2\left(x^2_2\right)^{-1} S[x^1,x^2]\right]\right) = 0,$$

$$\begin{aligned}\widehat{R}^3[x,p] = p_1 - \quad & M[x^1,x^2]\Big\{ \left(x^2_2\right)^2 \left(x^1_2 x^2_{tt} - x^2_2 x^1_{tt}\right) + \left(x^2_1 N[x^1,x^2] + x^1_2\right) x^2_2 p_2 \\ & + \alpha\left[x^2_2 x^1_{22} N[x^1,x^2] - x^2_2 x^2_{12} - \left(x^1_2 N[x^1,x^2] + x^2_1\right) x^2_{22}\right] \Big\} = 0,\end{aligned}$$

where

$$N[x^1,x^2] = \left(x^1_2\right)^2 + \left(x^2_2\right)^2, \quad M[x^1,x^2] = N[x^1,x^2]^{-1}\left(x^2_2\right)^{-2},$$

$$S[x^1,x^2] = \left(1 + x^1_2 x^2_1\right).$$

- The leading derivatives: $\{x^1_1,\ x^2_{11},\ p_1\}$.

Point symmetries

Theorem

The two-dimensional Mooney-Rivlin equations are invariant under an infinite-dimensional group of Lie point transformations given by the infinitesimal generators

$$\mathrm{R}^1 = \frac{\partial}{\partial t}, \quad \mathrm{R}^2 = \frac{\partial}{\partial X^1}, \quad \mathrm{R}^3 = \frac{\partial}{\partial X^2},$$

$$\mathrm{R}^4 = X^2 \frac{\partial}{\partial X^1} - X^1 \frac{\partial}{\partial X^2}, \quad \mathrm{R}^5 = x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2},$$

$$\mathrm{R}^6 = F_1(t) \frac{\partial}{\partial x^1} - F_1''(t)\, x^1 \frac{\partial}{\partial p},$$

$$\mathrm{R}^7 = F_2(t) \frac{\partial}{\partial x^2} - F_2''(t)\, x^2 \frac{\partial}{\partial p},$$

$$\mathrm{R}^8 = F_3(t) \frac{\partial}{\partial p},$$

$$\mathrm{R}^9 = t \frac{\partial}{\partial t} + X^1 \frac{\partial}{\partial X^1} + X^2 \frac{\partial}{\partial X^2} + x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2},$$

where $F_1(t)$, $F_2(t)$, and $F_3(t)$ are arbitrary functions of time.

- Maple/GeM computation
- Evolutionary forms and Noether theorem: next slide.

Cheviakov, A. and St. Jean, S. (2015).
A comparison of conservation law construction approaches for the two-dimensional incompressible Mooney-Rivlin hyperelasticity model. *JMP* 56, 121505.

Some references

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`GeM` for `Maple`: a symmetry/conservation law symbolic computation package.
`http://math.usask.ca/~shevyakov/gem/`

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